MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(9), em2497
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/14981

Undergraduate students' abstractions of kinematics in differential calculus

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Received 03 May 2024 ▪ Accepted 05 August 2024

Abstract

When undergraduate students learn the application of differentiation, they are expected to comprehend the concept of differentiation first, make connections between particular constructs within differentiation and strengthen the coherence of these connections. Undergraduate students struggle to comprehend kinematics as a rate of change in their efforts to solve contextual problems. This study sought to explore undergraduate students' construction of connections and the underlying structures of these relationships as they learn calculus of motion. The action-process-object-schema and Triad theories were used to explore undergraduate students' construction of connections in differentiation and the underlying structures of these relationships as they learn the calculus of motion. This study was qualitative which involved a case study of 202 undergraduate mathematics students registered for a Bachelor of Education degree. Data were collected through an individual written test by the whole class and semi-structured interviews with ten students purposively selected from the class. The interviews were meant to clarify some of the responses raised in test. The findings revealed that students' challenges in differentiating the given function were insignificant, but they need help to make connections of differentiation to its application to kinematics. Furthermore, students' coherence of the connection among displacement, velocity and acceleration was weak, coupled by their failure to consider the point when the object was momentarily at rest (which is central in optimization). The results of this study have some implications for instructors. The teaching of calculus and other 456 mathematical concepts should connect to the real-life application of those concepts so that 457 students can make meaningful interrelationships thereof. Kinematics for differentiation paves way for kinematics under the application of integration hence students' optimal conceptualization is of utmost importance.

Keywords: application of differentiation, APOS, derivatives, kinematics, triad, undergraduate students

INTRODUCTION

In South Africa, secondary school learners are introduced to limits and continuity in grade 12, which in turn paves way for the introduction to differential calculus. In differential calculus, learners are taught differentiation from first principles, rules of differentiation and practical problems involving optimization and rates of change (Department of Education, 2011). Learners' firm understanding of calculus concepts is a foundation to further studies in advanced mathematics at university (Brijlall & Ndlovu, 2013). Many studies have been conducted in both secondary and undergraduate students' understanding of derivatives in the South African context (Brijlall & Ndlovu, 2013; Maharaj, 2013; Siyepu, 2013) and the results demonstrated that students could apply competently the rules of differentiation to derive any given functions. However, studies have also shown that students grapple with the real meaning of derivatives, which are closely tied to rates of change (Desfitri, 2016; Lam, 2009; Tyne, 2016). The use of carefully selected contexts can help students in providing purpose, relevancy and meaning to differentiation concepts and skills (Doorman et al., 2022). The contexts can show the relevance of differentiation and support students in

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Contribution to the literature

- This study sought to explore undergraduate students' construction of connections and the underlying structures of these connections as they learn calculus of motion.
- The findings revealed that students' challenges in differentiating the given function were insignificant, but they need more help to make connections of differentiation to its application to kinematics.
- The results build up on the teacher professional knowledge and provides valuable data for further research on exploring how optimization problems can be solved.

building mathematical knowledge on their intuitions. In calculus, the contexts relate to the application of derivatives to realistic or artificial real-world situations (Mkhatshwa, 2023b). The use of contexts involves modelling activities where rates of change play an important role.

Optimization involves the process of maximizing or minimizing a modelled mathematical objective function by taking into consideration input variables and constraints (Alshqaq et al., 2022). It is one of the key contexts of the application of derivatives, which has wide applications in engineering, economics, manufacturing and marketing. To enhance students' proficiency in solving optimization problems, rates of change serve as a fundamental concept (Retamoso, 2022). Rates of change are basically derivatives, which measures the change of the output variable in respect to the change in the input variable. To differentiate is to find the gradient function, which represents the slope of a function at a particular point in the domain of function.

Kinematics is an instance of optimization, which is the study of one-dimensional motion that uses derivatives to interpret instantaneous displacement, velocity and acceleration concepts (Hitier et al., 2022). Kinematics sits at the intersection of calculus and mechanics, as a case of a notion "where the contribution of mathematics and physics cannot be separated" (Pospiech, 2019, p. 3). In kinematics, contextual problems foster the concepts in students' understanding of real-life motion of physical bodies, which in turn provides relevance, meaning and purpose to mathematical skills and concepts (Doormanet al., 2022). Contextual learning support students' construction of mathematical knowledge based on intuition as students develop and connect representations in the physics of motion. According to literature, students manage the general techniques of differentiation but fall short of the conceptual underpinnings necessary to explain procedures in contextual problems (Brijlall & Ndlovu, 2013; Mkhatshwa, 2024). More specifically, students struggle to comprehend kinematics as a rate of change (Talib et al., 2023). This study focusses on the use of derivatives to solve one-dimensional kinematics problems. Thus, this study sought to explore undergraduate students' development of the connections of derivatives to motion in one direction and the coherence of these connections. This leads to the research question: "What are undergraduate students' understanding of the application of derivatives to motion in one direction?" The motivation of this study was that undergraduate students' competence in solving optimization problems of objective functions is problematic (Mkhatshwa, 2023a). Kinematics in differentiation relies on optimization of the displacement function. This study's significance is that the results builds up on the teacher professional knowledge and provides valuable data for further research on exploring how optimization problems can be sassed (Matindike & Makonye, 2023).

LITERATURE REVIEW AND THEORY

Understanding Derivatives

To show understanding of derivatives, students should understand the rules of differentiation for specified functions, as well as scale word problems on the application of differentiation. The learning of techniques of differentiation has been sufficiently investigated and with the conclusion that students generally do not find performing procedures problematic. A study by Siyepu (2013) explored undergraduate students' errors as they learn derivative of functions in a chemical engineering course. After using the action-process-object-schema (APOS) theory to classify and interpret undergraduate students' errors, Siyepu (2013) discovered that students' greatest error was over-generalization of rules and properties of the power rule. Otherwise, students did not have specific challenges with differentiation techniques. It is the application of derivatives in contextual problems that requires students to exert more cognitive load.

Fatmanissa et al. (2019) note that contextual problems are considered difficult as they require the skill to make sense of the problem and transform them to mathematical representation, in addition to understanding the derivative aspect. Transforming context problems into mathematical symbols is not an easy feat for most students; oftentimes, students prefer to find numerical solutions almost directly from the numbers given at face value. Students jump into calculations without careful understanding of what is given in the problem, at the expense of deriving required functions (Wijaya et al., 2014). In the study by Fatmanissa et al. (2019), this problem was posed to students: "*A bullet is shot upward vertically. The relation between its height (h) in meter and time (t) in seconds with* $0 \leq t \leq 60$ *is defined by* $h(t) = 300t - 5t^2$. *Explain the bullet's height 5 seconds before, exactly at, and 5 seconds after its maximum height.*" A sample of 69 grade 11 students attempted the aforementioned problem but struggled to explain the trajectory of the bullet; rather they went ahead and substituted $t = -5, t = 0$ and $t = +5$ into the function as an attempt to solve the problem. There was a conceited focus on symbols than the phrase "its maximum height". Students are prone to difficulties in translating contextual problems into mathematics functions and subsequently decide on which function to use.

The study by Klymchuk et al. (2010) also posed a problem where university students were lost in translation. Students were given this problem to solve a trucking cost problem: "*The cost of running a heavy truck at a constant velocity of v km/h is estimated to be* $4 + \frac{v^2}{200}$ *dollars per hour. Show that to minimize the total cost of a journey of 100 km in the truck at constant velocity, the truck should run approximately 28 km/h.*" Only ten out of 197 students managed to derive the total cost function that was supposed to be minimized. Without the total cost function, all the 187 students could not meaningfully proceed to solve the problem. However, the same students performed well in evaluating explicit derivatives that required only mathematical manipulations and techniques. In some cases, students would try to simplify the question by making random substitutions of the numerical values given in the question but without understanding. In another study, Ellis and Turner (2002) used the graphical approach to develop students' conceptual understanding of kinematics. Students were in a good position to understand motion in a practical and learner-centered environment. On the other hand, Mkhatshwa (2023a) focused on the opportunities provided by some American textbooks for students to learn optimization. His findings revealed that there exists a relationship between opportunities to learn optimization provided by textbooks of calculus and the known students' difficulties when solving optimization problems.

The APOS Theory

Dubinsky (1991) proposed a type of research to determine the extent to which theories of learning and teaching mathematics can help mathematics educators to understand the learning process. Theories provide explanations of phenomena that can be observed in students who attempt to construct understanding of mathematical concepts. The knowledge gained from such research is useful for mathematics educators to suggest directions for future pedagogy that can be of value in the learning process of mathematical concepts. The theory by Dubinsky (1991) later came to be known as the APOS. With this type of research, there is no strict separation between an instructor and a researcher; they both share a common goal and participate in activities that would facilitate learning mathematics (Oktaç et al., 2019). Other theories were also used in some studies in mathematics education, like the three worlds (Tall, 2004) and the triad by Piaget and Garcia (1989). A mix of two theories in a single study was observed in some studies, for example, the APOS and triad were used by Dubinsky and McDonald (2001) and APOS and three worlds were used by Bilondi and Radmehr (2023).

The development of the level mathematics understanding of the students culminates in the schema for that mathematical concept. The APOS theory advocates that students learn a mathematical concept through a hierarchical development of action, process and object mental structures by organizing them in a schema. The APOS theory is explained in detail in the works by Arnon et al. (2014) and Asiala et al. (1996). According to the APOS Theory, a mathematical concept is first understood at the action level, which represents a set of explicit step-by-step instructions construction of knowledge. Evaluating derivatives of given functions following a specific technique is evidence of an individual at the action-level conception of differential calculus. When an action is repeated and reflected upon, an individual can make internal mental constructions called processes. An individual at the process level can perform the same actions but based on internal stimuli, whereby he or she can predict, work in reverse and work mentally. An individual attains the process conception of differentiation when he or she can reflect on their actions to determine the appropriate technique of differentiation of a given function and then differentiate the function either physically or mentally.

An object conception occurs when an individual perceives a process as a totality and he or she can perform actions and processes to this totality. The application of differentiation is situated in this knowledge domain whereby an individual identifies an object implicitly or explicitly and then computes the critical values when the event is optimized. In this study, an individual student should be able to evaluate the time when the particle is momentarily at rest and the respective distance, speed and acceleration when such occurs. Finally, a schema of a mathematical concept is a collection of many actions, processes and objects and other previously held schemas that need to be organized coherently so that an individual can decide the appropriate mental constructions required to be applied in order to deal with a given problem situation. This coherent framework provides a means to determine which phenomena are in the scope of the schema and those that are not. The application of differentiation has many contexts and students with a full schema development should be in a position delineate and solve appropriately.

The references to differentiation based on each of the four mental structures stated above constitute the genetic decomposition (GD), which forms part of the theoretical analysis of the mathematical concept. A GD is a description of testable predictions that constitutes a description of the actions, processes and objects of mathematical concept. If the postulates are constructed in a certain way by a generic student, then this student will be successful in using the mathematical concept to solve problems involving the concept (Dubinsky & McDonald, 2001). The researcher drafts the GD based on his knowledge of research and understanding of the mathematical concept. After using the theory to analyze and explain how fitting the data is to the theory, revisions to the GD and the enhancement of the theory may become necessary. The revised GD may result in another cycle of theoretical analysis and implementation of instruction, however, an enhancement of the APOS theory may lead to a better understanding of the construction of knowledge in that concept. The enhancement of APOS was done by incorporating the triad theory by Piaget and Garcia (1989), which led to a profound and deeper understanding of the development of schemas and provide better explanation of students' construction of knowledge. To determine the possible mathematical understanding of students in this study, a mix of the Dubinsky's (1994) APOS and Piaget and Garcia's (1989) triad frameworks was used.

The triad theory focusses on mental constructions that goes on in the mind of students when they learn a mathematical concept. According to the triad theory, before a schema becomes coherent, it must go through the intra-stage, inter-stage and trans-stage. The intra-stage is the preliminary level of conceptualization whereby a concept is thought of in an isolated manner in terms of its properties. An individual's actions, processes and objects are in isolation from other mental structures of a similar nature (Borji & Martínez-Planell, 2020). At this lowest stage of schema development, students would have an isolated collection of rules for differentiating specific functions. Once relationships and transformations are established between objects and other previously held schema, the individual is at the inter-stage. The schema development enters the inter-stage when students make connections between the nature of differentiation and its relationship to the science of motion. In the trans-stage, students begin to form coherent structures that underlie the relationships developed in the inter-stage. There now exists a robust connection between differentiation and its application to the science of motion. Being able to take appropriate decisions to solve optimization problems depicts the development of the trans-stage. When students' mental constructions develop to the trans-stage, it can properly be referred to as a schema. The APOS theory's description of schema coincides with the trans-stage of triad since that is where coherent structures emerge. The isolated (intra-) and connected (inter-) objects constitute a pre-schema. This represents the full schema development of application of differentiation to motion in one direction, as advocated by Skemp (1962), who defined schema as an organized body of knowledge that can be used to bear upon problem situations.

Table 1. The frequencies of students' written responses according to the preliminary analysis

Response type	Frequency
Blank	9
Incorrect	32
Partially-correct	159
Correct	2

METHODOLOGY

By conducting a qualitative methodology study following the case study approach, the researchers explored first-year Bachelor of Education mathematics-major students' understanding of the application of differentiation to motion in one direction. According to Yin (2009), a case study approach is an in-depth investigation of a single and well-defined contemporary phenomenon within its context. Data were collected through 202 students' written responses to a formal test administered on kinematics. The sole item for consideration in this study was given as, "*The motion of a particle from the origin O is described by the equation* $s(t) = \frac{2}{3}t^3 - \frac{17}{2}t^2 + 21t$, *where* s *is the distance in meters and* t *is the time in seconds. Find the acceleration of the particle when it is momentarily at rest, given that the time taken is at least 5 seconds.*" The test was written after the teaching the topic of application of differentiation using traditional instruction. A preliminary analysis of data categorized students' responses into blank, incorrect, partially and correct and the frequencies thereof were evaluated. Furthermore, a content analysis of the written responses was done to reveal the stages of students' concept development of kinematics in accordance with the APOS and the triad theories. The students' written responses were assigned terms K1, K2 and so on until K202 for ease of reference and confidentiality.

FINDINGS

Preliminary Analysis

The preliminary analysis of data for the 202 written responses revealed the frequencies shown in **Table 1**. The next sub-section elaborates on the content analysis to report the nature of the students' understanding in each category. Interview transcriptions are also included as a follow-up the students' written responses.

From **Table 1**, nine students left the question un-answered. Of these, five only copied the question only (see **Figure 1**) and four skipped the question entirely.

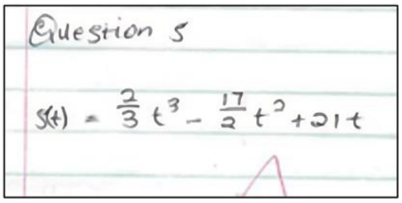

Figure 1. Writing down the question only by K93 (Source: Authors' own illustration)

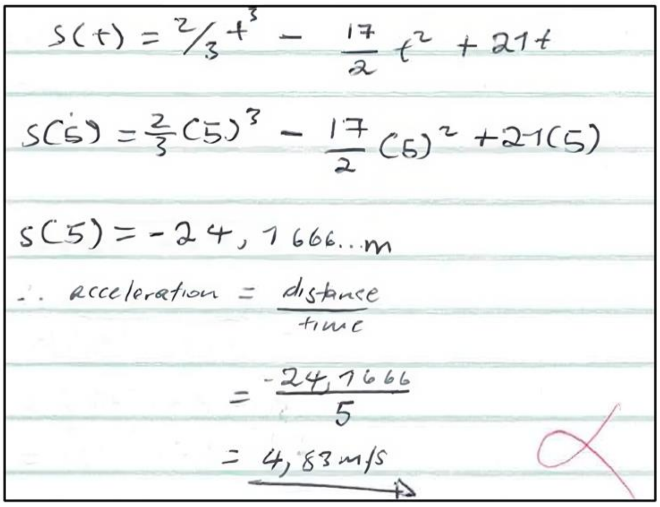

Figure 2. Failure to find the derivative of the polynomial $s(t)$ by K175 (Source: Authors' own illustration)

These nine students did not attain the intra-stage by not attempting, as they failed to recognize the question as the application of differentiation to the science of motion.

Incorrect Responses Category

The main challenge for the students in the category of wrong answers was the inability to differentiate the given displacement function. Students lacked the action conception to compute the derivative, $s'(t)$. Rather, students like K175 at once evaluated $s(5)$ to get distance and then used the unfounded formula $\frac{distance}{time}$ to find acceleration (shown in **Figure 2**).

If would have been better if he had used the correct formula $\frac{velocity}{time}$, but still how he was going to evaluate acceleration when the particle was at rest.

Of the 32 who got this question wrong, 17 did not differentiate the function at all in order to get the velocity function $s'(t)$. When asked to explain, K135 said, "*There is no mentioning of differentiation in the question.*" Unbeknown to K135, it is the velocity function that is used to get the time when the displacement is optimized. A further 13 did some attempts to differentiate but failed to get $s'(t)$ and/or $s''(t)$. K137 failed to get the first derivative as shown in **Figure 3**, which spoiled his chances of getting the expected values of time when the particle was stationary. The second derivative was not evaluated either in some cases.

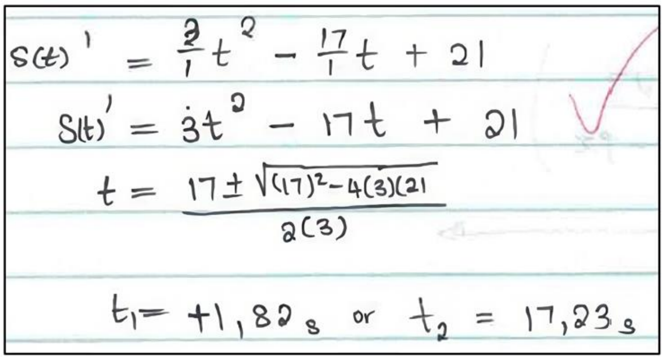

Figure 3. Incorrect first derivative by K137 (Source: Authors' own illustration)

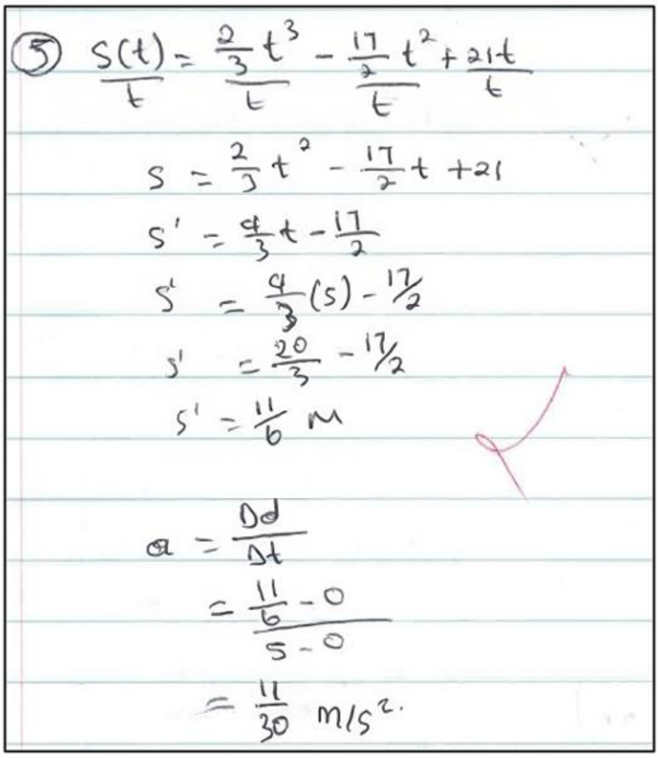

Figure 4. Inappropriate division by a variable t by K36 (Source: Authors' own illustration)

In another instance, K137 changed the polynomial to $\frac{3}{2}t^2 - \frac{17}{2}t + 21$ instead of the original function $\frac{2}{3}t^3 - \frac{17}{2}t^2 + 21t$. The remaining two students also changed the value of the function by dividing throughout by t and then differentiated the simplified function as shown in **Figure 4**. The student's intention to divide by t was unfounded.

Upon further inquiry in the interview, K104 had no specific reason to justify division by t as in the dialogue below.

> Researcher: Have you ever done this way when differentiating polynomials?
>
> K104: I do not remember. What I know is that we divide distance by time to get velocity.

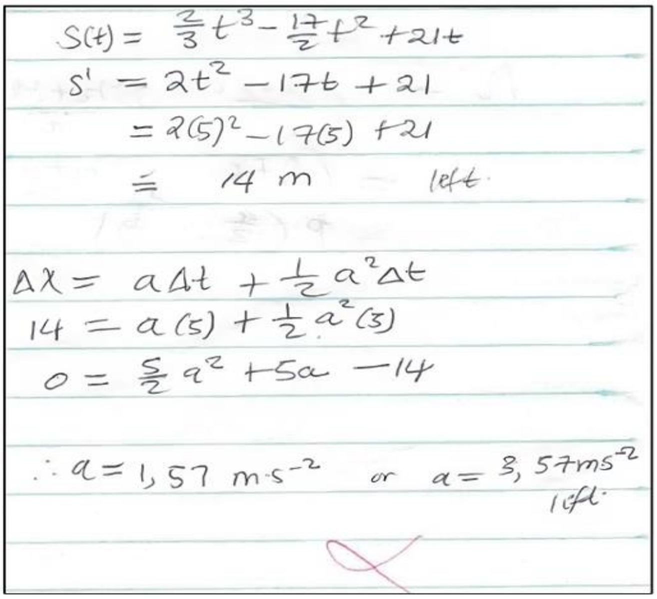

Figure 5. Use of equations of motions instead of derivatives by K97 (Source: Authors' own illustration)

Researcher: Is that the reason you divided by t?

K104: Yes.

Researcher: Ok. However, I see you still differentiated the quotient to get $s'(t)$.

K104: That was my understanding when I wrote that. However, I see my mistake now.

Sometimes students think if the rate of change is time-based, then division by time (t) is justified. Nevertheless, students who could not differentiate the given polynomial were insignificant relative to those who did differentiate correctly.

Having failed to get the correct derivative, the 32 could not make the required action conception of derivatives, and subsequently their connections of the application of differentiation to kinematics was weak. Students who majored in physics encounter kinematics under mechanics, where they also use equations of motions to evaluate unknown quantities like acceleration or velocity. Thus, two students tried to use equations of motions to evaluate acceleration, but it was out of context in optimization. **Figure 5** illustrates K97's attempt. Optimization uses instantaneous rates of change, where differentiation is key.

In the interview, K188 concurred that they covered kinematics in mechanics by using equations of motion. He confessed ignorance that equations of motion were inappropriate in this circumstance, insisting that was the only way to find the answer.

Partially Correct Responses Category

The highest frequency in the preliminary analysis was that of students who had partially correct responses.

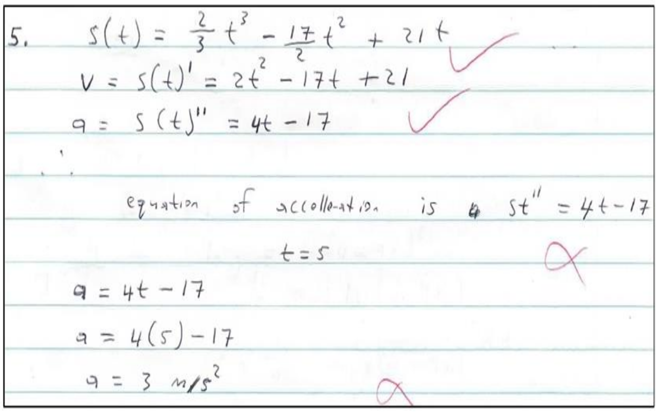

Figure 6. Correct differentiation but without sound connection to kinematics by K15 (Source: Authors' own illustration)

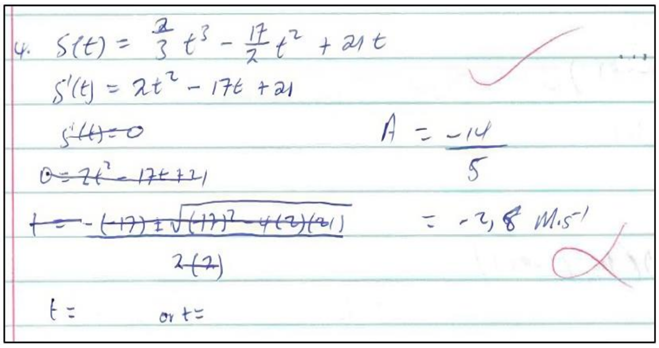

Figure 7. Correct first derivative but incorrect method for acceleration by K103 (Source: Authors' own illustration)

Of these, 55 students managed to establish the connection between derivatives and the three terms of kinematics, which are displacement, velocity and acceleration. However, they did not engage the idea of the particle being momentarily *at rest* to compute the critical time when such happens. This is illustrated in **Figure 6**, where K15 differentiated twice to get velocity and acceleration, respectively. However, he did not compute the critical times, so he had to put up with $s''(5)$ to find acceleration.

All the 55 students used the cut-off point of 5 minutes as the critical time to evaluate acceleration. To K145 he thought that $t = 5$ is the time when the particle was at rest, hence he got acceleration of 3 ms^{-2}.

Researcher: Did you use the concept of optimization to find time t.

K145: No Sir. It was given as five, wasn't it?

Researcher: What is your understanding of the term "at rest"?

K145: Oh, I was supposed to solve $s'(t) = 0$ first.

In a similar manner, 57 students differentiated only once and calculated $s'(5)$ to get $-14\,m/s$ (shown in **Figure 7**).

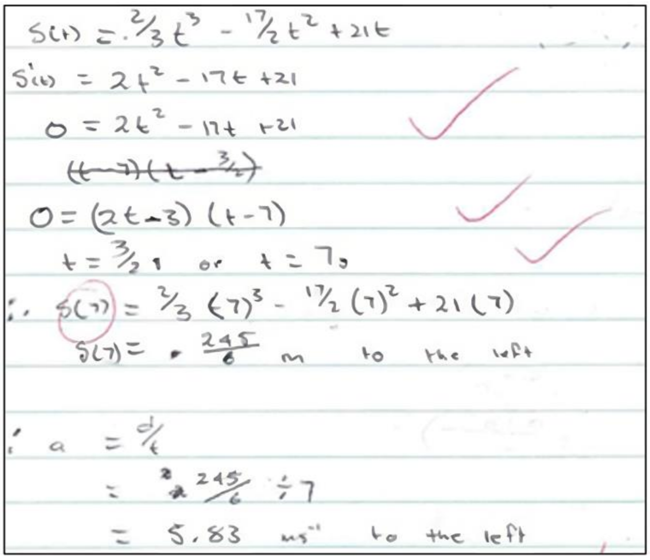

Figure 8. Correct critical times but without second derivative by K54 (Source: Authors' own illustration)

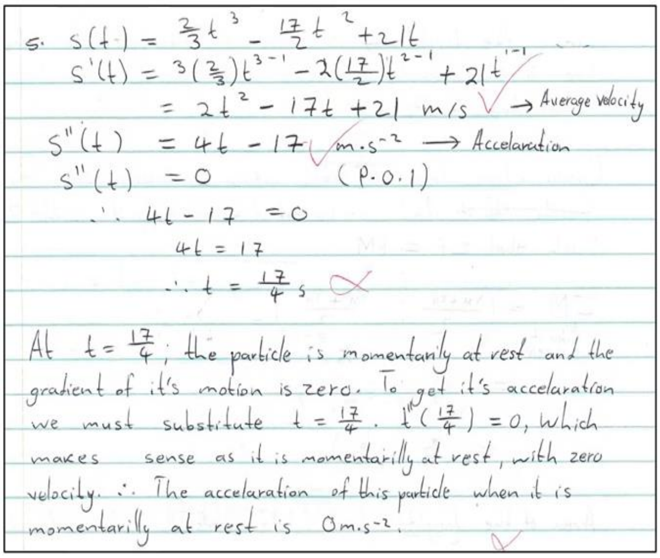

Figure 9. Using the second derivative to find stationary values by K20 (Source: Authors' own illustration)

Based on this velocity, they went ahead to evaluate acceleration as $\frac{-14}{5}\ m/s^2$. They conceived acceleration as average velocity divided by time, not as a rate of change requiring the use of the second derivative. Inadvertently, K103 cancelled the correct approach $s'(t) = 0$. When asked why he cancelled a supposedly correct procedure, K103 said it was not necessary since $t = 5$ was given.

A further 29 students made significant strides to get the solutions for $s'(t) = 0$ but did discard the solutions $t = \frac{3}{2}$ so as to evaluate $s''(7)$. Rather, there was a myriad of substitutions, like $s\left(\frac{3}{2}\right)$, $s''(3)$, $s(5)$, $s(7)$ and $s''\left(\frac{3}{2}\right)$. In **Figure 8**, K54 managed to solve the equation $s'(t) = 0$ and discarded the unwanted time but then plugged $t = 7$ into the function $s(t)$.

Again the second derivative was not evaluated, hence, acceleration was computed using $s'(t)$. However, K54 used $a = \frac{s}{t}$, rather than $\frac{v}{t}$. Their connection between instantaneous differentiation and the motion in one direction was muddled. After evaluating the first derivative, five students attempted but failed to solve $s'(t) = 0$. Solving the quadratic equation that has factors is supposed to be easy for undergraduate students; however, K12 tried to use completing the square but aborted the process as became lengthy. K103 also tried to use the quadratic formula to solve the equation $s'(t) = 0$ but later cancelled all the steps he had done. Four more students (K161, K201, K12, and K48) attempted to solve $s'(t) = 0$ but did not find the roots. Upon further inquiry in the interview, the students did not understand why they had to solve $s'(t) = 0$, hence were not keen to find the roots.

Exactly nine students missed the connection of derivatives to stationary values as evidenced by using $s''(t) = 0$, instead of $s'(t) = 0$. According to students like K20 (see **Figure 9**), optimization is tantamount to $s''(t) = 0$. K20 was quite convinced that the particle was momentarily at rest when $s''(t) = 0$ from his response.

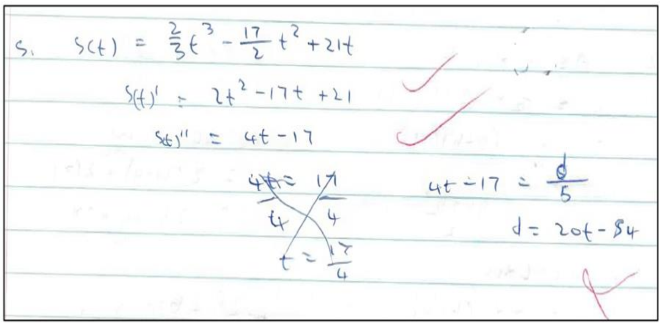

Figure 10. Perfect derivatives but incomplete solution by K53 (Source: Authors' own illustration)

When K20 was probed, he still maintained his explanation in **Figure 9**, oblivious to the fact that when a particle is momentarily at rest, the gradient is zero. This translates to $s'(t) = 0$ in symbols. Solving the second derivative equation gives rise to t=174, as shown in **Figure 9**. To find acceleration, K20 substituted $t = \frac{17}{4}$ into the function $s''(t)$ that obviously yielded zero. Three students also performed both derivatives perfectly but then ended the solution without calculating the stationary values. **Figure 10** portrays K53's solution that was not complete.

Correct Responses Category

From **Table 1**, only two students (K18 and K92) managed to establish the coherence of the relationship of acceleration when the particle was momentarily at rest. This entailed evaluating the second derivative and plugging in the expected critical time when the particle was momentarily at rest. The correct solution of the question is shown in **Figure 11**.

The dialogue with K92 is shown below:

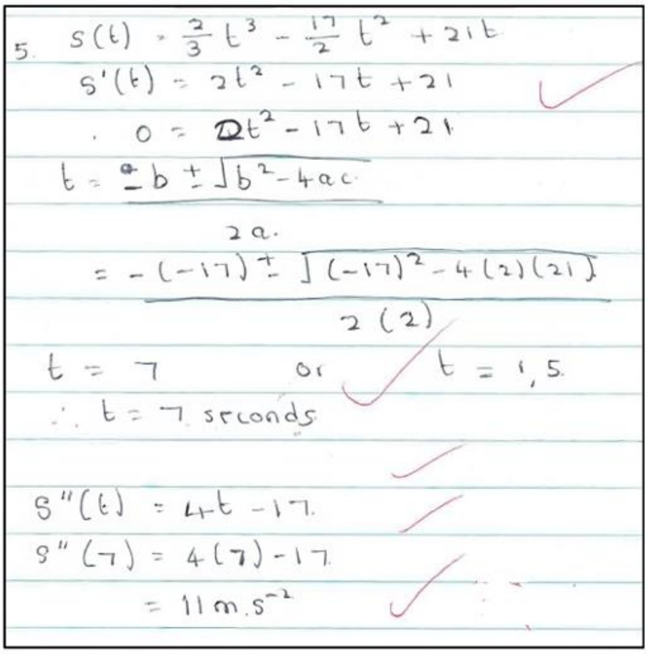

Figure 11. Perfectly correct solution by K18 (Source: Authors' own illustration)

> Researcher: Explain how you got the acceleration of 11 m/s/s.
>
> K92: The gradient is zero when the particle is momentarily at rest thus I solved the equation $s'(t) = 0$.
>
> Researcher: What happened to the root 3/2?
>
> K92: It is outside the solution set.
>
> Researcher: Why did you differentiate again?
>
> K92: Because acceleration is the second derivative in kinematics. Plugging in the critical time gives me acceleration when the particle is at rest.

The goal of instruction is to get the majority of students to at least develop the mental structures to evaluate critical values and subsequently differentiate once and twice for velocity and acceleration, respectively.

DISCUSSION

Evaluating Derivatives

In any given differentiation situation, some students opt to skip some questions. Researchers cannot tell whether the students could at least perform the appropriate techniques to find derivatives. This non-attempt represents a pre-schema according to Kazunga and Bansilal (2020). The findings in this study also revealed that students do not have particular problems in understanding and executing derivatives. The basics of calculus commence in the final year of high school in most countries, hence differentiation is a progression to first-year undergraduate calculus. Thus, high school teachers still have an enormous role to play to instill basic understanding of calculus concepts (Estonanto & Dio, 2019; Klymchuk et al., 2010). As is expected at the intra-stage, the students' understanding of derivatives was adequate, but it was isolated and rule-bound. Calculation of derivatives is oftentimes straightforward for most students, but applying differentiation is difficult when solving rates of change (Mkhatshwa, 2023a). Mkhatshwa (2023b) notes that the lack of rules of differentiation serves as a stumbling block for students when solving rates of change problems. Literature also reports on some perceived students' difficulties in evaluating derivatives (Brijlall & Ndlovu, 2013; Kertil & Dede, 2022; Tyne, 2016) but these are insignificant, as confirmed in this study. In this study, the given polynomial $s(t)$ was easily differentiated by the majority of students. Many students did not evaluate the first and second derivatives mainly because they did not realize that it was required to do so. The command to differentiate was implied, hence students like K135 did not differentiate because it was not explicitly stated.

Application of Differentiation

Students' weak connections in the application of differentiation are rife. Oftentimes "failure to express meaningful ideas on the optimization concept's role in calculus may to a large extent be due to inappropriate and weak mental links between knowledge of other calculus concepts such as derivatives" (Bezuidenhout, 2001, p. 23). Consequently, the findings in this study revealed that 32 students in the sample did not attribute the question to kinematics despite some conceited efforts to differentiate the given function. Klymchuk et al. (2010) comment that students are weak in establishing links between mathematics content and the real world, which in this study, are links between differentiation and its application in kinematics. Even though the term velocity was missing entirely in the item for the test, some key terms like distance and acceleration were present as cues to the question's connection to the motion in one dimension. The coherence of connections in a schema is determined by its use in deciding its scope in solving problems (García et al., 2011).

Equations of Motion

Some students in the sample were majoring in mathematics and physics. Since kinematics lies at the intersection of mechanics and calculus, some students resorted equations of motion to find acceleration. Instead of understanding the application of differentiation in totality, they resorted to the equations of motion as taught in mechanics. Motion in one direction is the same in both but what the students failed to realize is that calculus uses functions to describe motion, as opposed to mechanics which uses specific values that can be substituted into the set of equations of motion (Hitier & González-Martín, 2022). Moreover, kinematics in calculus considers instantaneous rates of change that leads to some values to be optimized, whilst the equations of motion rely on average rates of change that do not use derivatives. Such disconnections demotivates students, often leading to high failure rates amongst calculus students. The schema for the application of differentiation was under-developed in this case, which coincided with lack of inter-stage conception.

The Coherence of Relationships

Indeed students demonstrated the actions and processes in evaluating derivatives and linking them to kinematics. However, the trans-stage was not attained, which calls for establishing coherences in the connection they would have constructed in the science of motion. Students who attempted to solve $s''(t) = 0$, was evidence that they did not understand that in rates of change, it is the first derivative that is considered (Longe & Maharaj, 2023). To those who managed to solve $s'(t) = 0$, very few regarded the restriction that time should be greater than five minutes. Students in this category managed to establish the application to differentiation but did not get as far as the conclusion of the question (Fatmanissa et al., 2019). The articulation of acceleration when the particle was at rest require object-level conception of application of differentiation. The connection between the actions, processes and objects was partly achieved. Thus, it is difficult for students to develop object conception under the trans-stage of schema development. Bilondi and Radmehr (2023) also discovered that it is difficult to develop the object conception under the formal world. The formal world is the highest level under Tall's (2004) theory of three worlds and the most difficult to develop. The successful structural organization of actions, processes and objects is what Dubinsky (1991) defined as the schema. In this study, findings revealed that only two students attained the full schema development of the kinematics.

Students' challenges in establishing robust relationships to optimization tasks are to handle the problem assumptions and restrictions. The task in this study required acceleration when the particle was momentarily at rest, but the time was not given. Students had to implicitly compute critical times and then consider only time that is at least 5 minutes. In attempting to solve contextual problems, some students tend to pick numbers in a context and plug them into formulae. Fatmanissa et al. (2019) reveal that students pay more attention to numerals than to the phrase "momentarily at rest". The findings revealed that students regarded the phrase *at least 5 minutes* as denoting that time is given and it is five. Students are quick to jump to computing numerical solutions directly from the information they see without carefully constructing algebraic expressions from what they read in the problem (Wijaya et al., 2014). Furthermore, Hitier and González-Martín (2022) confirm students plug quantities into equations and churn out numerical solutions without due understanding of their calculations. Hashemi et al. (2015, p. 227) sum up by saying that "difficulties in learning derivatives and integrals among undergraduate students are due to their weakness in solving problems involving these concepts".

CONCLUSION

It was not easy for students to attain the trans-stage, which entails that the full schema development in kinematics as an application of differentiation was hardly achieved. Results in this study should be an alert to possible lack of schema development in undergraduate students' conceptualization of the application of differentiation. The teaching of calculus and other mathematical concepts should connect to the real-life application of those concepts so that students can make meaningful interrelationships thereof. Learning of a mathematical concept is complete only if students can apply what they would have learnt to real-life experiences and draw a strong coherence of understanding the concept and its application (where possible). Maharaj (2013) posited that growth in understanding derivatives hinges on establishing connections and the underlying relationships to the connections, between the mathematical representation and the physical application. In this study, students had no serious challenges with differentiation of polynomials, but they had formidable challenges to connect it to its application in kinematics. This was caused by over-reliance on isolated facts and procedures in the process of learning kinematics. This study argues that connections within–and interactions with–and coherent to other (similar) concepts are fundamental to meaningful learning. The implication to instruction is that growth in understanding contextual problems in derivatives depends on establishing coherent connections between a mathematical representation and a real-life application.

Funding: No funding source is reported for this study.

Ethical statement: The author stated that the study was approved by Walter Sisulu University on 10 December 2020 (Approval code: FEDSRECC01-12-20).

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

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