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The work of a prospective high school teacher in pre-professional training in highlighting mathematical knowledge

Paula Verdugo-Hernández [1*], Sofía Caviedes Barrera [2]

[1] Escuela de Pedagogía en Ciencias Naturales y Exactas, Facultad de Ciencias de La Educación, Universidad de Talca, Linares, CHILE

[2] Universidad Internacional de La Rioja, SPAIN

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Abstract

This article aims to identify the knowledge that enables a prospective secondary school teacher to mobilize number sense components during a pre-professional practicum. Using a qualitative case study approach, four classroom episodes focused on fractions, percentages, and algebraic language are analyzed. The study involves one prospective teacher and 23 students from a multi-grade school in central-southern Chile. The analysis integrates teacher's specialized knowledge and mathematical working spaces to explore how these frameworks inform the teacher's knowledge. The findings reveal that the teacher employs personal theories, real-life examples, and graphical aids to enhance students' understanding, though there is a reliance on rules and algorithms. While specialized knowledge aids in teaching, there is less emphasis on fostering students' conceptual understanding and judgement. The study recommends improving teacher training programs to better equip prospective teachers in grasping number sense components, highlighting the need for more focused training on its development.

Keywords: number sense, mathematics teachers´ specialized knowledge, mathematical working spaces, prospective secondary teachers

INTRODUCTION

Teacher education plays a crucial role in shaping the effectiveness of classroom instruction, particularly in mathematics, where specialized knowledge is fundamental for fostering students' mathematical understanding. Prospective teachers must develop both content knowledge and pedagogical strategies during their pre-professional practicum, which is essential for their future teaching practices. Numerous studies have established that the quality of mathematics instruction significantly depends on teachers' mastery of subject matter and pedagogical content knowledge (PCK) (Ball et al., 2008; Shulman, 1986; Strutchens et al., 2017). Furthermore, the connection between teachers' knowledge and student achievement has been a persistent theme in educational research (Alex, 2019; Mapolelo & Akinsola; Pournara et al., 2015; Roberts-Hull et al., 2015).

In Chile, as in other countries, mathematics education has been the subject of extensive policy reforms aimed at improving teacher preparation. Despite these efforts, gaps remain in the development of teachers' knowledge for teaching mathematics, particularly in areas such as number sense. Interpreting numerical situations across diverse contexts enhances comprehension, critical analysis, and application of mathematical concepts in various social, cultural, and professional environments. This capability is intricately connected to number sense, which is widely acknowledged as fundamental within mathematics education (National Council of Teachers of Mathematics [NCTM], 2000). The development of number sense begins in primary education, focusing on the recognition and representation of numbers, as well as understanding their function and applicability in everyday life (Maghfirah & Mahmudi, 2018). However, the transition from primary to secondary education introduces more complex number concepts, including whole numbers, fractions, decimals, and percentages, which require advanced number sense skills (Australian Education Council, 1990; Ministerio de Educación y Formación Profesional [MEFP], 2022). Despite its

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✉ paulasinttia@gmail.com **(*Correspondence)** ✉ sofialuisa.caviedesbarrera@gmail.com

Contribution to the literature

- This study integrates the Mathematics Teacher's Specialized Knowledge (MTSK) and Mathematical Working Spaces (MWS) frameworks to offer a comprehensive analysis of how a prospective secondary teachers mobilize number sense components in real classroom settings.
- The integration of the MTSK and MWS frameworks highlights how the prospective teacher's specialized knowledge shaped his teaching strategies within his suitable MWS, effectively bridging theoretical knowledge with practical application in the context of number sense. This approach offers a valuable lens to analyze the practical applications of teacher knowledge.
- The findings reveal that while the prospective teacher effectively uses graphical and symbolic representations, indicating proficiency in the number identification component, there remains a gap in developing the making judgement component, pointing to the need for practical improvements in teacher training programs.

importance, number sense development in secondary education remains an area of concern. Research shows that prospective secondary school teachers (PSTs) often lack the depth of number sense needed to meet curricular objectives, and their number sense is generally lower than that of primary teachers (Almeida et al., 2016; Wulandari et al., 2020; Yang et al., 2009). This gap hinders their ability to foster students' mathematical reasoning and problem-solving abilities, particularly in more advanced topics where students are expected to transition from procedural fluency to deeper conceptual understanding. This study contributes to the ongoing research on number sense and teacher education by addressing a gap in the existing literature and offering practical recommendations for enhancing teacher preparation programs. By focusing on secondary education, where number sense becomes increasingly important, the findings of this research provide valuable insights into improving the practical application of theoretical knowledge in educational settings, thereby better supporting students' mathematical development.

Objective

This article aims to identify the knowledge that enables a PST to mobilize number sense components during a pre-professional practicum, focusing on the relationship between teacher knowledge and its application in secondary education. By integrating the mathematics teacher's specialized knowledge (MTSK) framework (Carrillo et al., 2018) and the mathematical working spaces (MWS) theory (Kuzniak, 2011), this study explores how these theoretical models inform the teaching practices of prospective teachers, providing valuable insights into the translation of specialized knowledge into classroom instruction. Pre-professional practicums offer critical opportunities for PSTs to apply theoretical knowledge in real classroom contexts (Blömeke et al., 2014), but the effectiveness of these experiences relies on the quality of their pedagogical knowledge and ability to foster number sense development (Clark & Peterson, 1986).

THEORETICAL FRAMEWORK

In the subsequent section, we will delineate the components of number sense and the theoretical constructs of MWS and MTSK that form the foundation of this study. By integrating these theories, we aim to present a comprehensive framework that facilitates an in-depth exploration of both mathematical practices, and the specialized knowledge required for effective teaching. This approach will also elucidate the development of number sense by examining the intricate relationships and interactions between these theoretical perspectives.

Number Sense

The NCTM (2000) states that number sense is one of the fundamental ideas of mathematics, as it enables students to

(1) understand number, ways of representing numbers, the relationships between numbers and the number system,

(2) understand the meanings of operations and how they relate to each other, and

(3) calculate fluently and make reasonable estimations (p. 32).

Conceptual understanding is crucial for number sense, as it allows students to connect numbers and operations, thereby solving problems in flexible and creative ways (Markovits & Sowder, 1994, p. 23).

Number sense has been defined in terms of various components by different authors. For example, McIntosh et al. (1992) distinguish three components: knowledge of numbers, knowledge of operations, and the application of this knowledge to computational environments. Building on this categorization, further research identifies additional components for defining number sense (Ghazali et al., 2021; Şengül & Gülbağcı, 2012; Yang et al., 2008). The approach of Ghazali et al. (2021) was selected for this study due to its comprehensive breakdown of number sense components, which stems from an extensive literature review on number sense. Its relevance to current educational practices has been highlighted, and it has been proven useful for exploring teacher knowledge and the associated development of number sense in secondary school students (De Gamboa et al., 2024). The components of this approach include:

(1) number composition, which involves understanding the structure and decomposition of numbers,

(2) number identification, encompassing the recognition and transition between different representations of numbers (e.g., visual-verbal, verbal-visual, verbal-manipulative), for example, in the case of rational numbers, it is necessary to understand when it is more convenient to use each representation (Charalambous & Pitta-Pantazi, 2007),

(3) magnitude of number, focusing on the comparison and ordering of numbers, considering absolute and relative magnitudes, symbols, and position on the number line, and including misconceptions associated with whole numbers and decimals (Resnick et al., 1989),

(4) arithmetic operations, involving mental calculation and basic operations (McIntosh et al., 1992), along with dispelling misconceptions about multiplication and division, and

(5) judgement, which relates to the reasonableness and accuracy of calculations (Yang et al., 2009), such as understanding that dividing a number by 2 is equivalent to multiplying by ½.

The development of the above-mentioned components in students requires teachers to have a strong number sense, enabling them to relate whole and rational numbers (fractions, decimals, and percentages), apply strategies to solve problems, and understand the meaning of numbers in real-world contexts (De Gamboa et al., 2024; Gay & Aichele, 1997). However, research indicates that PSTs often lack robust number sense, as they tend to prioritize rules and algorithms for problem-solving and struggle with estimating quantities (Markovits & Sowder, 1994; Reys & Yang, 1998). For instance, Mawaddah et al. (2021) assessed various number sense components in 81 Indonesian PSTs, revealing a low mastery of basic number system concepts, particularly visual and abstract representations, despite showing better accuracy in calculations. Similarly, Alajmi and Reys (2007) examined how practicing PSTs in Kuwait evaluate the reasonableness of students' answers. They found that PSTs often equated the reasonableness of an answer with numerical accuracy and did not incorporate the reasonableness component in instructional planning or activity design, likely because it was not included in the national curriculum. Additionally, Almeida et al. (2016) compared the number sense of Spanish PSTs with primary school teachers in Taiwan (Yang et al., 2009). The study showed that Spanish PSTs had lower number sense, particularly in using reference points for ordering rational numbers (magnitude of number). The reasoning approaches also differed: Spanish PSTs relied more on number sense components, whereas Taiwanese primary teachers tended to use rule-based reasoning.

These findings highlight the need for improved training and development in number sense for PSTs, ensuring they can effectively teach and apply these concepts in their classrooms. Teacher knowledge plays a crucial role in the development of students' number sense, as research indicates that teachers with a strong number sense are better able to facilitate students' understanding of numerical concepts (Almeida et al., 2016; De Gamboa et al., 2024). However, many PSTs often exhibit limited number sense, heavily relying on rules and algorithms (Markovits & Sowder, 1994; Reys & Yang, 1998). Recognizing the impact of teachers' knowledge on their ability to effectively teach number sense is essential for advancing mathematics education.

Mathematics Teachers´ Specialized Knowledge

The MTSK theory serves as an analytical tool to interpret and analyze teacher knowledge across various domains and subdomains, enabling the mapping of their expertise (Carrillo et al., 2018). This theory comprises two major knowledge domains: mathematical knowledge (MK) and PCK (**Figure 1**). MK pertains to the understanding of mathematics as a scientific discipline within an educational context, distinguishing between mathematics as an academic field and school mathematics (Carrillo et al., 2018). Within MK, there are three subdomains, with the study focusing on knowledge of topics (KoT). KoT involves a deep

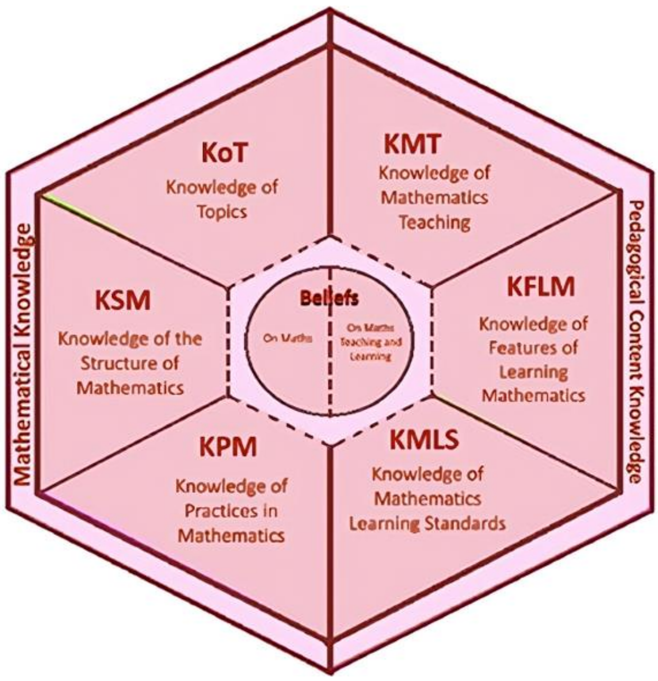

Figure 1. Domains and subdomains of MTSK model (Carrillo et al., 2018)

comprehension of mathematical content and its meanings, recognizing the complexity of mathematical objects that may arise in the classroom. This subdomain includes concepts (e.g., rational numbers), definitions (e.g., what constitutes a rational number), procedures (e.g., multiplication of fractions and decimals), representations (e.g., visual, symbolic), justifications (e.g., when to use fractions or decimals), intra-conceptual connections (e.g., relationships between concepts, definitions, justifications, etc.), as well as facts and theorems. A strong foundational KoT is essential for teaching number sense effectively, ensuring that PSTs can accurately and comprehensively convey mathematical concepts to students.

PCK relates to the knowledge of mathematical content in the context of learning and teaching. This study considers two subdomains within PCK: knowledge of mathematics teaching (KMT) and knowledge of the features of learning mathematics (KFLM). KMT encompasses the MK that informs teaching practices, incorporating both personal and institutional theoretical knowledge specific to mathematics education. This includes designing learning opportunities and teaching strategies for topics such as fraction, percentage and number sense components. Effective teaching strategies are crucial for facilitating students' understanding and application of number sense, enabling them to grasp complex concepts and procedures. KFLM pertains to the understanding of how mathematical content is learned, focusing on the learning processes students must undergo to grasp various content elements, such as using symbolic notation to divide decimals and fractions. Additionally, KFLM considers the emotional aspects of learning mathematics, such as students' anxiety during problem-solving, as well as their motivations, mathematical interests, and expectations. These factors can influence the choice of representations used when presenting a problem for a particular topic. Recognizing and addressing the learning processes involved in understanding number sense is essential for tailoring instruction to meet students' needs and enhance their learning experiences.

By integrating these domains and subdomains, the MTSK provides a comprehensive framework for analyzing and improving mathematics teaching, highlighting the critical aspects of teacher knowledge necessary for fostering student understanding and engagement. The selected subdomains specifically address the knowledge and strategies required to effectively teach and develop number sense, making them highly relevant to this study. This is particularly important for PSTs, who are in the process of developing the expertise needed to teach mathematics effectively at the secondary level.

Mathematical Workspaces

The theory of MWS allows for the analysis of mathematical work performed when solving a given task, considering both the cognitive and epistemological planes (Kuzniak et al., 2022). The epistemological plane comprises components such as referential elements (e.g., properties, theorems, definitions), representamen (e.g., semiotic signs), and artefacts (both material and symbolic). The cognitive plane includes components of visualization (e.g., spatial representation and material support), construction (depending on instruments and associated techniques), and proof (validation through discursive processes based on theoretical referential) (**Figure 2**).

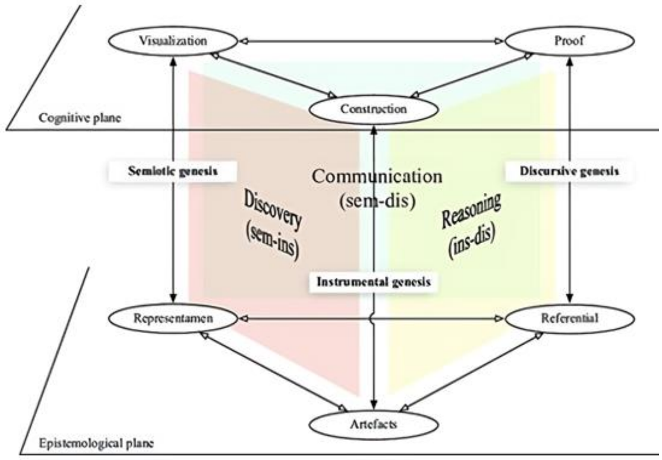

Figure 2. The mathematical working space (Kuzniak & Richard, 2014. P. 23)

The aforementioned planes are connected through three types of genesis (Gómez-Chacón et al., 2016): instrumental genesis, which is evident when certain artefact's (symbolic or instrumental) converge to an outcome, either towards the artefact or construction; semiotic genesis, which is evident when certain signs associated with semiotic representation converge to an outcome, either towards visualization or representamen, or vice versa; discursive genesis, which is evident when certain definitions or properties associated with the discursive converge to an outcome, either to proof or to referent, or vice versa. In the MWS, gene interactions activate three vertical planes (Kuzniak et al., 2016): semiotic-instrumental [Sem-Ins], where artefacts aid in building or exploring representations; instrumental-discursive [Ins-Dis], focusing on proofs through experiments and justifications; and semiotic-discursive [Sem-Dis], which integrates proof with visualization.

Kuzniak (2011) outlines three types of MWS: reference MWS, personal MWS, and suitable MWS. The reference MWS depends on mathematical organization established and defined on mathematical grounds. Several research studies (e.g., Gaona et al., 2023) have focused on initial teacher training under the study of the suitable MWS which deals with the adequacy and organization of reference MWS, with the purpose of making it feasible to implement in an educational institution. Finally, personal MWS is about how an individual performs mathematical work based on his or her knowledge and cognitive abilities (Kuzniak, 2011). In this paper, we delve deeper into the suitable MWS implemented in the classroom.

Networking MTSK-ETM

Complementarity between theories is understood as a refinement produced by the elements of each framework used when analyzing a single phenomenon (Gómez-Chacón et al., 2016). In this sense, the process of linking theories is not reduced to the triangulation of different theoretical perspectives to improve understanding (Bikner-Ahsbahs, 2009), but to develop systematic tools to connect theories, theoretical approach and use of theory (Bikner-Ahsbahs, 2009). This complementarity, as a research practice, has been developed in recent years under the concept of "networking" (Bikner-Ahsbahs & Kildron, 2015; Bikner-Ahsbahs & Prediger, 2009; Prediger et al., 2008a, 2008b). There has been an increasing interest in possible relations between theories (Haspekian et al., 2023), where both MWS and MTSK models have been shown to be potential fields of study based on this connection of theories (Castela, 2021).

The MTSK theory provides a comprehensive framework for analyzing teacher knowledge, encompassing both MK and PCK (Carrillo et al., 2018). Meanwhile, the MWS theory offers a valuable perspective for examining the cognitive and epistemological aspects of mathematical work (Espinoza-Vásquez et al., in press; Kuzniak et al., 2022). Integrating these theories can deepen our understanding of how teachers mobilize number sense in the classroom (Gómez-Chacón et al., 2016).

Combining MTSK and MWS allows for a detailed examination of the interplay between teacher knowledge and teaching practices. MTSK focuses on the specialized knowledge that teachers possess, while MWS elucidates how this knowledge is enacted through the use of representations and artefacts. This integration addresses both the content of teachers' knowledge (MTSK) and its application in practice (MWS). By merging these models, we achieve a holistic analysis of the knowledge base, and the cognitive processes involved in teaching number sense. This integrated framework supports a richer analysis of how teachers facilitate the development of number sense, enhancing both educational theory and practice.

METHOD

General Characteristics of the Research

This study follows a qualitative, interpretive approach using a case study design (Bryman, 2009) to explore the mobilization of number sense during pre-professional practicum sessions. These are structured, supervised sessions that are a critical component of teacher education programs, where PSTs apply their theoretical knowledge in real classroom environments. The practicum involves responsibilities such as lesson planning, instruction, student interaction, and classroom management, all under the supervision of a mentor or supervising teacher. In this study, one PST taught mathematics to a multi-grade classroom of 23 students in central-southern Chile, focusing on topics such as fractions, percentages, and algebraic language. These sessions provided valuable opportunities for the PST to mobilize number sense components and refine teaching skills. The practicum not only allowed the PST to implement pedagogical theories but also served as a formative experience for enhancing specialized knowledge and classroom management. Video-recorded classroom sessions and group discussions between the PST and students were analyzed. Non-participant observation (Cohen et al., 2007) was employed during the video recordings, with researchers interpreting discursive interactions to identify how number sense components were mobilized, categorizing them according to the MTSK-MWS frameworks.

Case Selection

In a national project investigating the practicum experiences of PSTs, eight PSTs from the same public university in Chile were initially involved. However, only three PSTs obtained consent for video recording during their pre-professional practicum due to limitations with parental and student approvals. One of these PSTs, referred to as "Mario," was selected for further analysis. Mario is in his fourth year of training, having completed most mathematics subjects, two didactics subjects, and three pre-professional practicum subjects. His upcoming practicum will be his final one before the professional practicum. The choice of Mario as the focal PST was primarily driven by the ease of accessing data (Loughran et al., 2008). Six video-recorded class sessions on fractions, percentages, and algebraic language were conducted with Mario and a group of 23 students aged 12-13 from a multi-grade school (7th grade and 8th grade) located in central-southern Chile. The sessions, each lasting approximately 90 minutes, covered topics such as representing and operating with rational numbers, solving word problems involving rational numbers, and using algebraic language.

Data Collection and Analysis

The analysis follows the guidelines of network theory (Bikner-Ahsbahs & Kildron, 2015) as an analytical tool to integrate different theoretical perspectives. Rather than applying multiple frameworks independently, network theory allows for a deeper connection between the

Table 1. Analysis protocol for number sense components (adapted from Ghazali et al., 2021)

Number sense components	Descriptors
Number composition	Meaning of numbers.
Number identification	Recognition of numbers in their different representations; strategic use of particular representations.
Magnitude of numbers	Order and comparison of numbers; magnitude, relative and absolute magnitude, position of numbers on the number line.
Arithmetic operations	Strategic use of operations, mental arithmetic.
Make judgement	Validation of the accuracy of results and procedures

Table 2. Protocol for MWS analysis (Henríquez-Rivas & Verdugo-Hernández, 2023)

Criteria	Components	Descriptor
Semiotic genesis (SG)	Representation	Relates mathematical objects and their significant elements.
	Visualization	Interprets and relates mathematical objects according to cognitive activities. Related to registers of semiotic representations.
Instrumental genesis (IG)	Artefact	Use of material artefacts or a symbolic system.
	Construction	Based on the processes given by the actions triggered by artefacts used, and the associated techniques of use.
Discursive genesis (DG)	Referential	Use of definitions, properties or theorems.
	Proof	Discursive reasoning is based on different forms of justification, argumentation or demonstration.
Vertical plane	[Sem-Ins]	Artefacts are used in the construction of results under certain conditions, or for the exploration of semiotic representations.
	[Ins-Dis]	The process of proof is based on experimentation with the use of an artefact or the validation of a construction.
	[Sem-Dis]	The process of visualization of the represented objects is coordinated with discursive reasoning in order to test it.

MTSK and MWS models. This approach enables a more comprehensive analysis of the interplay between teacher knowledge (Authors., in press) and its application in classroom practice, particularly in relation to number sense. It offers a more nuanced approach to studying teacher knowledge and its impact on learning environments. In line with the above, the analysis proceeds through five phases:

(a) Researchers collectively decide which data to analyze, selecting classes and episodes that offer a rich variety of examples and representations for analysis. These episodes are chosen because they illustrate the PST's mobilized knowledge within the MTSK framework and their ability to manage their suitable MWS. Each unit of analysis is identified through discursive interventions between the PST and students, which highlight the teacher's pedagogical strategies and content knowledge.

(b) Selected class recordings are transcribed, and content analysis is conducted (Bardin, 1996). Researchers analyze the data through their theoretical frameworks (**Table 1** and **Table 2**).

(c) Researchers exchange and discuss their findings.

(d) Results are restructured considering points of agreement and disagreement.

(e) Following several meetings and analyses, a consensus on the final results is reached.

To further explore the study's focal point and adhere to guidelines (a) and (b), we have drawn inspiration from the approach outlined by Kuzniak and Nechache (2021). This involves initially providing a broad description of the class before progressing to theoretical analysis. Theoretical considerations relating to number sense, MTSK, and MWS are examined and interpreted in the following two moments:

Moment 1: Number sense and MTSK

The initial phase centers on identifying and categorizing the number sense components utilized by Mario (Ghazali et al., 2021), based on the following criteria:

Furthermore, we identify the discursive interventions (from both PST and students) that elucidate the knowledge mobilized by Mario, based on criteria described before.

Moment 2. Number sense and MWS

The key interventions where mathematical work components are activated are identified. Special focus is given to how Mario manages his suitable MWS while addressing number sense components.

Analysis of Classroom Episodes

Class 1 begins with four routine exercises related to operations with fractions. Subsequently, Mario proposes four-word problems for students to work on addition, subtraction, multiplication and division of fractions.

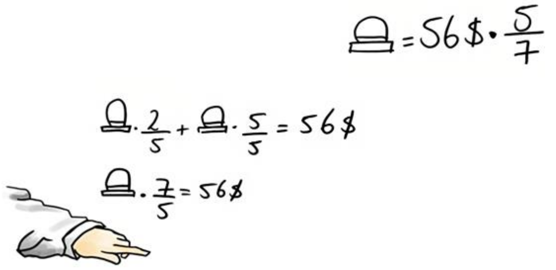

Figure 3. Solution illustrated by Mario (Source: Authors' own elaboration)

Class 1–Moment 1: Number sense analysis–MTSK

The number sense component pertains to arithmetic operations, focusing on solving operations involving fractions and multiplying by multiples of 10. Interventions from line 1 to line 9 illustrate Mario's mobilization by emphasizing the necessary procedures for fraction operations. This component aligns with specialized knowledge in the KoT subdomain, specifically involving procedures for adding fractions (such as equating denominators) and the concept of multiplicative neutral to facilitate these procedures (though informally mentioned). This suggests an approach to KFLM based on Mario's personal theories, as it implies his awareness of the challenges students face with adding fractions. Additionally, the mobilization of the KMT subdomain is evident when Mario says, *"what I can do is erase the suit and change it for the hat, for the same hat there, ok? multiplied by $\frac{2}{5}$, ok?"*, as he uses equivalent representations (**Figure 3**) given their potential when teaching addition of fractions.

> 1 Mario: […] The trousers plus the hat equals 56 dollars. Now, I don't know how much either one costs, I know the total, but I do know that the suit costs two-fifths of the hat. So, what I can do is erase the suit and exchange it for the hat, for the same hat right there, ok? multiplied by $\frac{2}{5}$.
>
> 2 Student: Yes.
>
> 3 Mario: Now […] we are only talking about hats, and we are looking at the price. So, I have $\frac{2}{5}$ of the price of a hat and the price of a hat. if I want to solve this, how can I do it? […] I'm going to show you another trick... to add fractions, what has to be there? do you remember? because addition is involved in multiplication […] That they have the same denominator, okay?
>
> 4 Student: Yes.
>
> 5 Mario: So, the trick I'm going to do here is to multiply the hat by 5 and divide it by 5, how much is it … [writes $\frac{5}{5}$].
>
> 6 Student: 5 divided by 5 ... 5.
>
> 7 Mario: 5 divided by 5 is 1 [...] If I multiply something by 1, does it change?
>
> 8 Student: No.
>
> 9 Mario: No, […] this trick what I've just done with this problem is to help me to be able to add these two, ok? these two prices [...] we only have the fractions 2 plus 5 [writes 7 in the numerator] and for the sum of fractions we keep the denominator [writes 5 in the denominator leaving $\frac{7}{5}$ = 56$ ... now, if $\frac{7}{5}$ here is being multiplied, how would I pass it to the other side ... I pass it by dividing, okay?

Class 1–Moment 2: Number sense analysis and MWS

The arithmetic operations component is inferred, encompassing fractions and multiplication by multiples of 10 (line 10 to line 16). In terms of MWS, Mario directs the class towards four problem-solving tasks focused on the division and multiplication of fractions within relatable contexts (Henríquez-Rivas & Verdugo-Hernández, 2023), including:

> "In a shipment 132 boxes of $\frac{47}{8}$ kg. each arrives, what is the total weight of all the boxes?", "Barbara always reads twice the number of pages of a book each day, twice the previous day. If on Monday she read $\frac{1}{7}$ of the number of pages of the book, in how many days will it be finished?

For example, in solving the shipment problem, Mario uses the concept of the multiplicative neutral as a symbolic artefact to facilitate fraction manipulation. This artefact allows students to solve the problem effectively. However, the use of the multiplicative neutral could be perceived as a 'trick' without proper theoretical justification, indicating a need for a more robust referential and precise arguments to activate the discursive genesis coherently and appropriately for the level at which Mario teaches. Furthermore, within his suitable MWS, Mario employs a calculator as an artefact to handle larger numbers in his calculations.

> 10 Mario: No, no, look, look at this magic trick [writes one in the denominator $\frac{32}{1}$] … ohhh Why? Because a number divided by one, what does that give me?
>
> 11 Student: One.
>
> 12 Mario: No, no. 132 divided by one, how much is it?
>
> 13 Student: 132.

Figure 4. Representation of percentages (Source: Authors' own elaboration)

> 14 Mario: 8 divided by one?
>
> 15 Student: 8.
>
> [...]
>
> 16 Mario: OK, [...] We use this magic trick for multiplication of fractions and division, ok? When we multiply a number that has no denominator by a fraction, we can do this trick, ok?

In class 2, Mario begins with a percentage example using a load bar. He then shifts focus to routine exercises and calculations, centered around division (and the corresponding standard algorithm) to determine specific percentages.

Class 2–Moment 1: Number sense analysis–MTSK

Interactions between Mario and students reveal the mobilization of various number sense components. The number identification component is evident as Mario employs various representations for percentages, enabling him to later operationalize a definition for this content. An instance of this occurs at the lesson's outset (line 17-line 18) when he remarks:

> 17 Mario: I imagine you know what a load bar is, don't you? […] when you download something a load bar appears […] So, it appears like this ... 1% already ... And when it gets here, 10%, right? [points to a part of the bar] Then after that ... 20%, 30% and it's increasing. So, what is that percentage telling you?
>
> [...]
>
> 18 Mario: So, it's going to reach 35, 40, 41, 42, 43, 44, 45 [...] 50, 50% downloaded. So, the percentages are numbers from 0 to 100 that represent the amount of something [...].

The above example shows that the number identification component is determined by the strategic mobilization of KoT-definitions. In other parts of the lesson, Mario presents a tangible representation for percentages (**Figure 4**), aiding students in visualizing how wholes are divided into parts and thereby understanding the abstract concept of percentages. An example of this is seen in the following excerpt (line 19 to line 26):

> 19 Mario: If I have this door [...] where is 50% of it, more or less around here, right? [points with his hand horizontally to the middle of the door] ... you have the table, where is 50% of it?
>
> 20 Student: [points to the middle of the table]
>
> 21 Mario: Perfect so 50% is the same as dividing into 2. You have the table I cut 2 pieces. But watch out, in equal pieces, OK? [...] I'm going to cut the door, I cut it here [points with his hand to one end of the door] [...]. Would it be 50%?
>
> 22 Student: No.
>
> 23 Mario: So, you take something, and you divide it in two [...].
>
> 24 Mario: Mmm let's say you have a birthday [...] you invite all your classmates [...] 19 people came, plus you are 20, right? you want to divide this cake [draws the cake] into 20 pieces ... a bit difficult, isn't it? First let's divide it in half, then in half again. How much do we have there?
>
> 25 Student: 4.
>
> 26 Mario: 25%, then each piece we have to divide it into 5. So, we have 1,2,3,4 and 5 [divides each quarter into 5 parts] more or less like that, right? Then we have 25, 25, 25 [pointing to ¾ of the cake] And here 5, 5, 5, 5, 5, 5, 5, 5, 5 [pointing to the ¼ divided earlier]. So, if I do the same with all the pieces, how many pieces will I have left? 5, 5, 5, 5, 5, 5, 5, 5, 20 pieces, right? and how many guests were there? [...].

Once more, it's inferred that the number identification component is tied to Mario's KoT, particularly through the use of diverse representations. Likewise, it's evident that Mario mobilizes KMT categories by strategically selecting representations based on their potential for teaching percentages. The number magnitude component is identified when Mario later employs a pencil and a drawn tape measure as a unit line to illustrate the positioning of percentages on that line (line 27-line 29).

> 27 Mario: When I say ... let's see your pencil, leave it here [points to the student's table] Where is half of your pencil?
>
> 28 Student: [points to the half of the pencil].

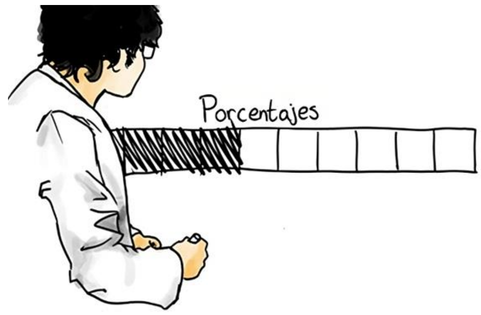

Figure 5. Representation of percentages in a load bar (Source: Authors' own elaboration)

> 29 Mario: Where's 25%? Where's a quarter of your pencil? That would be half of a half, right? [...] if I now tell you 10%, where would you put it?... you have to imagine the pencil and divide it into ten equal squares, where would it be? more or less? over there ... [student points to a place on the pencil].

The third component inferred is arithmetic operations, with Mario stressing the significance of mental calculation (basic multiplication and division operations) and the requisite procedures for obtaining percentages. For instance, he highlights *"if you want 50% divide by 2; if you want 20% divide by 5; if you want 5% divide by 20"*. Hence, this aspect is associated with the mobilization of KoT-procedures for percentage calculations. Similarly, Mario mobilizes KoT-Procedures to endorse techniques like shifting decimals when multiplying and dividing by powers of 10, as well as adding or removing zeros when multiplying and dividing by multiples of 10. For instance, he states:

> "What happens when we divide a number into 10? we have to remove a zero, right? […] If we have 100 students in the school, and 10% of them like to listen to rock music, how many students like to listen to rock music? Because that's how we do the questions … out of 100 of those students, 10% ... you take the 100 and divide it into 10".

Within this same component, an insight into Mario's KFLM emerges, as he acknowledges students' struggles with multiplication and division procedures and advocates for the use of fingers to aid in these calculations.

Class 2–Moment 2: Number sense analysis and MWS

Mario evidences a referential of percentages by consulting its definition and illustrating it using a load bar (**Figure 5**). This example mobilizes the referential aspect related to the magnitude of numbers component, reflecting semiotic genesis within the MWS framework. By visually and graphically representing percentages, Mario helps students transition from abstract numerical concepts to concrete visual forms, enhancing their understanding through semiotic processes. This approach characterizes his suitable MWS and evidences the flexibility he possesses in transitioning between the definition of fractions and their representation. Mario's use of these representations can be examined through semiotic genesis where the load bar serves as semiotic artefact that help students translate abstract percentages concept into tangible visual forms. This involves the representamen component of the epistemological plane, where students link mathematical symbols with their visual representations, thereby enhancing their understanding of fractions.

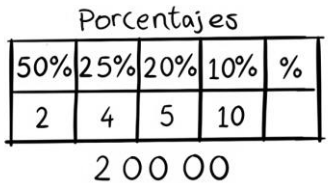

Figure 6. Table of values and percentages (Source: Authors' own elaboration)

It's evident that a hallmark of Mario's classes is his utilization of various examples, understood as that which represents something that helps students' generalization. These examples are employed as activating tasks in the class, including the following example of mental calculation and showcasing the activation of the arithmetic operations component due to its procedural nature.

> 30 Mario: If I want to know 50% of the students in this room how much is it? Fast!
>
> 31 Student: Ehh 22 … 50%.
>
> 32 Mario: 50%.
>
> 33 Student: 11.
>
> 34 Mario: 11, what did you do?
>
> 35 Student: I divided it.
>
> 36 Mario: By how much?

Furthermore, Mario presents various examples (**Figure 6**) supported by a table, indicating an activation of semiotic genesis as he represents percentage values in the table, facilitating visualization. This implies Mario's mobilization of the number identification component.

It's noted that the student's personal MWS lacks division of the assigned task, evident when Mario enquires about it:

> 37 Mario: In 10% of your life, how old are you?
>
> 38 Student: 3 years old.
>
> 39 Mario: You are 13 years old, the 10% of your life [...]. you have to divide it into 10 […].

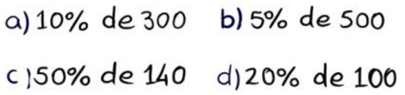

Figure 7. Type of task proposed by Mario in relation to percentages (Source: Authors' own elaboration)

> 40 Student: I don't know how to divide.
>
> 41 Mario: [...] you have to know how to divide, I'm going to have to send multiplication and division homework.

Mario proceeds with additional contextualized examples, showcasing a tendency for illustrating situations through drawings and an underlying presence of semiotic genesis. For instance, Mario utilizes the classroom door to exemplify a percentage, engaging visualization by allowing perception of stimuli through an object (as per Duval, 1999). Moreover, there's evidence of mobilization of the number identification component through a conversion from the visual to the numerical register of representation. Notably, Mario employs gestures (as described by Arzarrello et al., 2009) when dividing with his fingers, lo que evidencia la importancia que Mario le otorga a la representación, as he mentions: *There is no shame in occupying the fingers. I tell them 4 divided into 2, and they say 1, 2, 3, 4. I say, um... 9 divided into 3; 1, 2, 3. 1, 2, 3. 1, 2, 3. 1, 2, 3 [pointing with fingers].* It's noteworthy that the majority of the class is led by Mario, with occasional interactions with students facilitated through questions. In over half of the lesson, Mario assigns a task for the students to complete, involving percentage calculations (**Figure 7**). Mario does not introduce contextualized problems or other activities in the latter part of the lesson. The procedural nature of the tasks set by Mario suggests the mobilization of the arithmetic operations component.

Mario continues to teach percentages, highlighting four methods for calculating percentages of various quantities using multiplication and division:

(1) dividing the total by 100 and then multiplying by the percentage,

(2) multiplying the total by the percentage and then dividing by the total,

(3) multiplying the total by the decimal equivalent of the percentage, and

(4) using the rule of three.

Mario's examples in each instance are routine in nature. However, two-word problems requiring percentage calculations are also addressed.

Class 3–Moment 1: Number sense analysis–MTSK

Only the arithmetic operations component is evident, as Mario employs his KMT to underscore both mental calculation and step-by-step procedures for obtaining percentages. This is illustrated when he mentions:

> "[...] So, simple, it asks us for 23% of 450, we take the total, we divide it into 100, we multiply it by the percentage, right? Simple, right? [...] strategy number 2 tells us, for example, 15% of 300, OK? [he writes it on the whiteboard] [...] now we are going to multiply first. So, they tell us first we multiply 300 by 15, 300 by 15 how much does that give us?"

For both strategies Mario mobilizes knowledge linked to his KoT-procedures, albeit in a cursory manner. Likewise, the same emphasis is placed on introducing strategy three:

> 42 Mario: "[…] we are asked for the decimal equivalent to the percentage requested. This is the most direct way. For example, 36% of 2,400. So, percentages, as you know, are measured from 0 to 100, right? [...] to get the percentage of the number we can multiply it by the decimals. [...] So, if I want 50% of something […] I multiply it by 50 divided by 100, that is 0.5, and those numbers […] will always be <zero-point something> they will go from 0 to 1. So, if I want 10%, it's 0.1, if I want 50%, it's 0.5 [...] Perfect. Then it tells me, multiply the number, that is to say, the 2400 by the decimal equivalent to the percentage [36%]. So, it would be 2400 multiplied by how much?
>
> 43 Student: [...] it would be 0.36.
>
> 44 Mario: Perfect.
>
> 45 Student: It's just that I do it the other way, Mario [...] I put the 0.36 first and then I put the 2,400.
>
> 46 Mario: OK, both ways are correct […], it doesn't matter the order. In this case it's going to give us 864. OK?

In the above extract, Mario prioritizes procedural explanations by drawing upon his knowledge associated with KoT-procedures, KoT-properties, and KFLM. For instance, he employs the commutative property of multiplication to illustrate to a student that the order of factors doesn't impact the final product. This use of the commutative property is important for procedural fluency, but it can also result in student difficulties, as some students may struggle to understand why the order of multiplication does not change the result. Mario's awareness of these potential misconceptions reflects his KFLM, enabling him to address and clarify these during instruction. Mario also mobilizes knowledge of KoT-representations, acknowledging that percentages can be expressed as decimal numbers and emphasizing their equivalence. The final strategy Mario presents pertains to the simple rule of three, where the focus is procedural, stressing the importance of placing

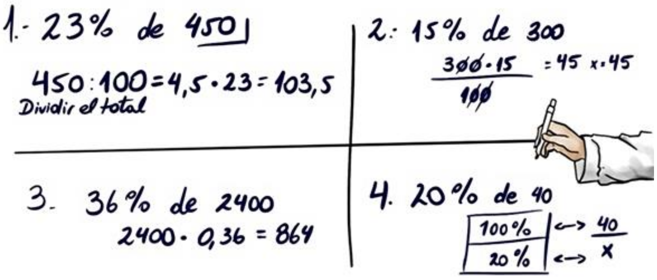

Figure 8. Calculation of percentages (Source: Authors' own elaboration)

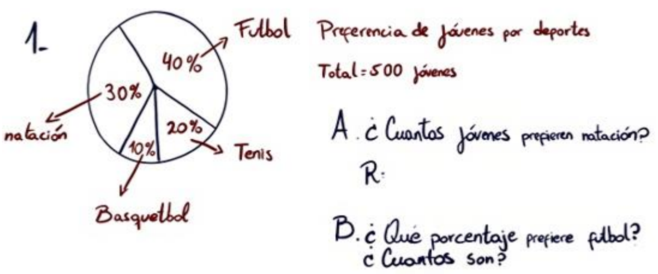

Figure 9. Representation of percentage in pie chart (Source: Authors' own elaboration)

percentages alongside quantities, without delving, for example, into the proportional relationship underlying the magnitudes involved. To validate his results, Mario employs a calculator.

When working on problem solving, Mario highlights the procedures for percentage calculation, stating that any of the four methods mentioned previously can be applied. However, the focus of the problem-solving work centers on the sequential steps to be followed, and the justifications that enable reasoning and judgement behind each resolution are missing. This is exemplified in the following explanation:

> 47 Mario: [...] half of the percentage who prefer tennis are men. So, how many women prefer tennis? we are going to assume that half of tennis is divided into men and women [...] what does it ask us first? the percentage that prefers tennis [...] what slice do we want? 20% [points to the pie chart drawn on the whiteboard]. So now the first thing we do is to calculate 20% of 500, right? but that's one step [...]. This answer needs […] two steps. Here we were asked directly, how many young people prefer football? So, we did one step [...] In this case it asks us something else, to calculate the percentage of tennis and then it asks us how many women prefer tennis?

Class 3–Moment 2: Number sense analysis and MWS

A referential regarding percentage calculation is recognized, encompassing four distinct procedures for this task (**Figure 8**). Hence, it's inferred that the above is linked to the arithmetic operations component. In the first procedure developed by Mario on the board (**Figure 8**), we notice an error when attempting to calculate 23% of 450. He indicates that 450 ÷ 100 = 4.5 × 22 = 103.5, which suggests an imprecise referential and understanding of the importance of equality and could lead to an incorrect personal MWS for the student. Additionally, in each of the strategies, the variable "x" is either not shown (strategy 1 and strategy 3) or appears only at the end (strategy 2 and strategy 4).

The four strategies are viewed as theoretical artefacts, activating the instrumental genesis (Kuzniak et al., 2016), as the algorithms outlined facilitate the necessary percentage calculations. Mario employs graphical representations like pie charts and load bars to explain percentages. These representations help students visualize the concept of percentages, making abstract ideas more concrete and easier to understand. Through the MWS lens, the use of pie charts and load bars can be examined through semiotic genesis as they act as semiotic artefact s that help students translate abstract percentage concepts into concrete visual forms. This involves the representamen component of the epistemological plane, where students connect mathematical symbols with their visual representations, enhancing their understanding of percentages.

By introducing the initial contextualized problem, Mario activates the semiotic genesis, employing a pie chart representation (**Figure 9**). This form of illustration enables visualization of the percentages in question, potentially aiding the students' calculations. Thus, it is inferred that Mario mobilizes knowledge linked to the number identification component, as there's a clear conversion between the symbolic percentage representation and the graphical depiction in the pie chart. Moreover, it's deduced that Mario solves problems as he knows or believes they should be solved, without adhering to a defined structure, implying that his suitable MWS may need supporting theories for his teaching.

Class 4 is marked by beginning with two illustrative examples, followed by Mario presenting a formal definition concerning algebraic language. Subsequently, attention is directed towards six statements, with students tasked to work on them individually before reviewing them collectively on the whiteboard.

Class 4–Moment 1: Number sense analysis–MTSK

Mario mobilizes knowledge linked to the number identification component, as he stresses the importance of employing algebraic language to represent expressions in everyday language. For instance, when Mario highlights

> "[...] I am 10 years older than a person [...] in algebra there is something very important which is the use of letters [...] I am 10 years older than a person, so that person it is going to be defined as

"A" [...] "A" is equal to the age of a person [writes A + 10] [...] and what is this? [points to A+10 and a student says double] No, it is not double, it is the age plus 10, OK? [...] the aim of algebraic language is not to use defined quantities, that is to say, we leave what comes next, what is known as an unknown, a number that we know",

he mobilizes knowledge associated with KoT-Representations to identify the equivalent expression to his verbal statement, albeit without emphasizing the significance of the unknown. Moreover, Mario mobilizes knowledge associated with KoT-Definitions by utilizing definitions for algebraic and natural language. For instance, he comments regarding natural language, *"[...] refers to words, written; and algebraic language refers to occupying letters and numbers"*.

Another example illustrating the number identification component is presented in lines 48-56. Here. Mario highlights the symbols used to articulate the symbolic representation corresponding to the verbal expression.

48 Mario: We have two numbers here that we don't know. [...] we are told that one third of the difference [...] how do I say that this is divided into three [points to (-) written on the whiteboard], how do I show it?

49 Student 1: I draw a line at the bottom and put the number on it.

50 Mario: Is that OK? a fraction? Would that be OK?

51 Student 2: No.

52 Mario: Why not?

53 Student 1: Because we can't subtract ... we can't divide, is that dividing?

54 Mario: Yes, we can. Remember fractions are divisions, aren't they?

55 Student 2: Ah right!

56 Mario: Get used to using fractions because it's much easier than putting the two dots to divide, ok? Kids, when you want half of something, let's say the number M, how much is half of M? [...] So, that's what you have to study, that's what you need to study a little bit, the words, ok?

The above extract also indicates the mobilization of knowledge linked to KoT-procedures, albeit centered on rules and steps necessitating adherence (or memorization) for converting verbal representations into symbolic forms. Furthermore, when Mario states, "*Get used to using fractions because it is much easier than putting the two dots to divide*", he mobilizes knowledge linked to the phenomenological approach to fractions as division and KFLM. For instance, when a student struggles with the concept of using fractions to represent division, Mario explains, "*Remember fractions are divisions, aren't they?*", though he does not explore the concept in depth, which might leave some challenges unaddressed for the students.

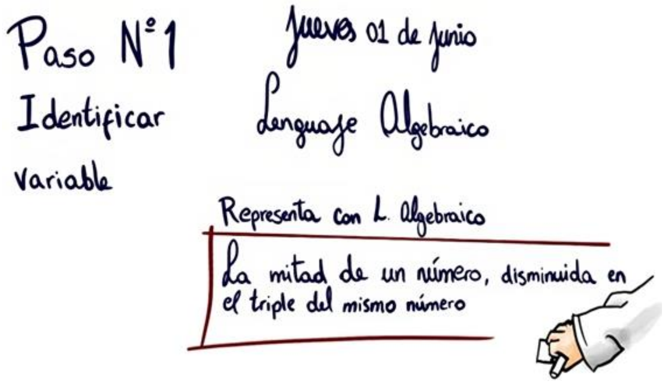

Figure 10. Example of task of algebraic language (Source: Authors' own elaboration)

Class 4–Moment 2: Number sense analysis and MWS

When presenting the topic "algebraic language", Mario resorts to definition-type examples, which activates the semiotic genesis that is subsequently mobilized towards the referential by means of the definition provided, as can be seen below:

57 Mario: […] I'm 10 years [...] older than a person, OK? I'm going to put it like this, [...] first of all we can write 10, right? Now [...] in algebra there is something very important which is the use of letters [...] we use a lot of letters, literary language, OK? […] I'm going to define that person as "A".

58 Student: And why "A" and not "P" as in [...]?

59 Mario: You can choose any letter, but I have to say down here [writes on the whiteboard] "A" is equal to the age of a person [A + 10].

The absence of an initial definition prompts the student to question the selection of the letter "A" over alternatives like "P". Following two examples, Mario offers an informal definition of algebraic language, stating *"[...] so algebraic language is not to use defined quantities, that is, we leave what comes, what is known as an unknown, a number that we know [...]"*. Then, he gives the definition of algebraic language when he highlights: *"Set of symbols that allow to generalize and formulate the calculations of arithmetic [...]"*, which characterizes the referential of his suitable MWS. In the subsequent part, Mario introduces a task wherein he outlines that they will examine the steps necessary for its completion (**Figure 10**). Among these, the initial step involves

Table 3. Number sense summary, components of MWS and MTSK

CLASS	Number sense from MTSK			Number sense from MWS			MTSK			MWS			VP/G
							MK	PCK		CP	EP		
	IN	SO	MN	IN	SO	MN	KoT	KMT	KFLM	V	R	A	
C1		x			x		x	x	x	x		x	[Sem-Ins]
C2	x	x	x	x	x	x	x	x	x	x	x	x	Sem. G
C3		x		x	x		x	x	x	x	x		Sem. G
C4	x			x			x	x	x	x	x		Sem. G

Note. IN: Identification of numbers; SO: Sense of operations; MN: Magnitude of numbers; CP: Cognitive plane; V: Visualization; EP: Epistemological plane; R: Representamen; A: Artefact; Ref: Referential; VP: Vertical plane; & G: Genesis

identifying the variables, followed by representing them in algebraic language.

During the development of the given task, discrepancies in the understanding of multiplication are apparent in the personal MWS of certain students, as they fail to grasp multiplication as a process of augmentation (in this context), as evidenced below:

> 60 Mario: [...] What would double something mean?
>
> 61 Student 3: Times 2.
>
> 62 Student 1: Half.
>
> 63 Mario: No, it's not half, it's multiplying it by.
>
> 64 Student 2: 2.

Throughout the explanations, it is evident that Mario relies on various representations and transformations of conversion, as demonstrated in the following excerpt where he writes m:3 and $\frac{m}{3}$, indicating a connection to the number identification component:

> 65 Mario: What did you get confused about?
>
> 66 Student: In the division.
>
> 67 Mario: [...] when we, for example, divide 4 into 2 [he writes $\frac{4}{2}$ it's the same as 4: 2] That's the same thing, ok. So, if I tell you half of 4, you can present it to me in these two ways, ok? [...] if I tell you one third of M, how would you write it? M divided by three [writes $\frac{M}{3}$ or writes M:3] [...] I'm asking you for a third of an operation of a difference. [...] There is a number that you don't know, subtracting with another number [...] we don't know what the numbers are, but what we do know, is that they are divided into three [writes $\frac{(-)}{3}$] OK? That was the first part, one third of the difference.

RESULTS AND DISCUSSION

Table 3 illustrates the knowledge Mario mobilizes across the four analyzed classes. His specialized knowledge reflects a certain level of number sense, with a particularly rich emphasis on procedural knowledge. This strong alignment with his individual KoT is evident in his reliance on procedures for conducting calculations. However, Mario places less emphasis on assessing or encouraging students' judgement when manipulating fractions, computing percentages, or working with algebraic language. This observation is consistent with findings from other studies showing that prospective teachers often struggle to move beyond procedural fluency to foster deeper conceptual understanding (Ball et al., 2008; Mawaddah et al., 2021). Despite this, Mario effectively employs a range of representations, such as graphical and symbolic notations, to facilitate student understanding. These representations are crucial for helping students connect abstract mathematical concepts with concrete visual forms, aligning with the MWS framework, particularly in terms of visualization and the semiotic genesis of mathematical concepts (Kuzniak et al., 2022). The discussion below explores these themes in greater detail by linking Mario's teaching strategies to the different components of number sense and his specialized knowledge.

Class 1: Fractions and Procedural Fluency

In the first class, Mario addresses the number identification component by intertwining his personal theories with teaching strategies to tackle students' challenges, such as using gestures to explain division difficulties (Arzarrello et al., 2009). This component is linked to the strategic use of representations (Ghazali et al., 2012; McIntosh et al., 1992), reflecting Mario's specialized knowledge and providing insights into the suitable MWS that Mario prioritizes. He introduces the concept of fractions by emphasizing procedural steps like finding common denominators for addition and subtraction. This approach highlights his KoT and his KMT through effective communication of these procedures. Additionally, Mario's ability to address students' difficulties shows his KFLM, as he uses his understanding of common student misconceptions to guide his teaching strategies effectively. For instance, when students struggle with understanding how to add fractions with different denominators, Mario uses an example involving the price of a hat and trousers. Mario's use of graphical representations, such as fraction bars, helps students visualize the process of finding common denominators, aligning with the cognitive plane of the MWS and its visualization component. By effectively employing both graphical and symbolic representations, Mario illustrates part-whole relationships and equivalency in fractions. These representations bridge the gap between abstract concepts and concrete understanding, facilitating a deeper comprehension of fraction addition and subtraction for students.

However, Mario's lack of focus on fostering the conceptual understanding and making judgement component–such as explaining why these procedures work–highlights a gap in his specialized knowledge. This aligns with other studies that emphasize the need for teacher training programs to balance procedural skills with conceptual knowledge development (Yang et al., 2009). While Mario effectively communicates procedural techniques through clear instructions and representations, students may miss opportunities to gain deeper insights into the meaning behind the procedures, limiting the development of their mathematical judgement skills.

Class 2: Percentages and Real-Life Applications

In the second class, Mario places a strong emphasis on the number identification component, particularly in the context of percentages, and effectively utilizes the magnitude of numbers component by using representations such as unitary lines to make abstract concepts more tangible for students, evidencing his referential of these components. By focusing on the procedural aspects of calculating percentages, including converting fractions to percentages and determining the percentage of a given number, Mario shows a solid grasp of procedural knowledge (KoT) and the referential aspects of the epistemological plane, showcasing his ability to effectively convey these procedures.

Mario's use of real-life examples, such as determining the percentage of a discount, helps students understand the practical application of percentages. This use of contextualized problems aligns with the discursive genesis, where Mario's verbal explanations and contextual examples guide students' reasoning processes and facilitate their understanding of percentages. Mario employs representations like pie charts and load bars to explain percentages. These graphical representations help students visualize the concept, making abstract ideas more concrete and easier to understand. there remains a heavy reliance on procedural fluency. While this strategy aids in practical understanding, it risks neglecting the development of conceptual understanding and making judgement, both of which are essential for fostering number sense. This observation aligns with findings from previous research, which shows that PSTs often prioritize rules and algorithms over deeper conceptual understanding (Almeida et al., 2016).

Mario's ability to connect abstract mathematical concepts with real-life applications reflects his while his ability to address students' difficulties reflects his KFLM, for instance, at the lesson's outset, Mario uses a load bar to explain percentages For instance, at the lesson's outset, Mario uses a load bar to explain percentages. Nevertheless, the results suggest that further training is needed to help PSTs transition from teaching procedural techniques to fostering critical thinking and number sense development in students, as noted by Roberts-Hull et al. (2015).

Class 3: Linking Percentages to Fractions and Decimals

In the third class, Mario continues to focus on teaching percentages, particularly by linking them to fractional and decimal equivalents while addressing the number identification component. His strategic use of representations evidences his ability to address students' challenges and clarify their understanding of numerical relationships. Mario emphasizes the procedural aspects of calculating percentages, such as converting fractions to percentages and finding the percentage of a given number. This approach highlights his KoT and referential of the topic, and his ability to convey these procedures effectively evidences his KMT. Moreover, Mario's KFLM is evident as he recognizes students' difficulties in understanding the order of operations and the equivalence of different mathematical expressions. For instance, when a student expresses a different method for multiplication, Mario validates the student's approach and clarifies that the order of multiplication does not affect the outcome. His approach highlights his proficiency in procedural knowledge but, once again, lacks sufficient focus on fostering students' judgement and reasoning skills.

Mario's use of real-life examples, such as determining the percentage of a discount, helps students understand the practical application of percentages. This use of contextualized problems activates the suitable MWS and aligns with the discursive genesis, where Mario's verbal explanations and contextual examples guide students' reasoning processes and facilitate their understanding of percentages. This indicates another characteristic element of Mario's suitable MWS. While Mario effectively uses contextualized problems and real-life examples, these primarily serve to reinforce procedural steps rather than encouraging students to develop a deeper understanding of the underlying mathematical concepts and the making judgement component.

Class 4: Algebraic Language

In the fourth class, Mario emphasizes the number identification component as he guides students in understanding algebraic language, using clear examples to help them grasp the relationships between numbers and algebraic expressions. His ability to translate verbal expressions into symbolic representations evidences his proficiency in mobilizing KoT and supports the semiotic genesis component of MWS. This process facilitates students' transition from verbal to symbolic representations, which is crucial in fostering mathematical abstraction. Mario introduces students to the use of variables and the structure of algebraic expressions, emphasizing the understanding of terms and coefficients. This approach reflects his KoT and his ability to convey these concepts evidences his KMT. Mario's ability to address students' difficulties shows his KFLM, as he recognizes students' difficulties in understanding the conceptual meaning of fractions as division and attempts to address this misconception directly.

Mario uses examples to activate semiotic genesis, which is subsequently mobilized towards the referential through provided definitions. For instance, Mario states, 'I am 10 years older than a person... in algebra, we use letters. I will define that person as "A".' He uses this example to illustrate the concept of variables in algebra, connecting everyday language to algebraic expressions. This process highlights the use of semiotic representations to transition from verbal to symbolic representations. Despite these strengths, Mario's reliance on rules and examples may limit opportunities for students to develop the judgement-making component.

CONCLUSIONS

This study aimed to identify the knowledge enabling a PST to mobilize number sense components during his pre-professional practicum. The findings underscore the critical importance of both procedural fluency and conceptual understanding in teaching number sense (Almeida et al., 2016; Markovits & Sowder, 1994; Reys & Yang, 1998). The results reveal that, unlike similar research (Mawaddah et al., 2021), Mario exhibits proficiency in number identification, as evidenced by his effective teaching strategies that transform symbolic representations into graphical or iconic forms. However, the findings also suggest that Mario may sometimes overlook the component of making judgement when preparing lessons and selecting tasks for students (Alajmi & Reys, 2007), indicating a possible area for further development in initial training (Pournara et al., 2015).

The integration of the MTSK and MWS enabled a more nuanced analysis of number sense components, considering both the PST's knowledge and student interactions. The MTSK theory has been instrumental in characterizing the essential knowledge components for teaching fractions, percentages, and algebraic language. It clarifies that teaching strategies based on the strategic use of representations (KMT) are closely linked to both the teacher's KoT and KFLM, allowing the teacher to address students' difficulties effectively (Caviedes et al., 2023). The MWS theory complements the MTSK by providing a lens to analyze the practical application of this knowledge in classroom interactions. For instance, when teaching percentages, Mario's strategic use of pie charts and load bars exemplifies the alignment between MTSK and MWS. His specialized knowledge (KoT and KMT) was evident as he used these representations to help students connect abstract concepts to concrete visual forms. This alignment is particularly clear in the semiotic genesis process, where students transitioned from understanding percentages in verbal terms to visual and symbolic representations. Mario's specialized knowledge influenced how he configured his teaching strategies within his suitable MWS, bridging theoretical knowledge with practical application.

The integration of theories shows that number sense, when viewed through the lenses of MTSK and MWS, may align or diverge regarding component types. For example, the magnitude of number and number identification components exhibit a connection with semiotic genesis, necessitating visualization and representation processes. Consequently, these components are explicitly manifested in MWS, such as linking a pie chart to the concept of percentage. By employing both theories, this study provides a comprehensive view of how specialized knowledge (MTSK) translates into effective teaching practices (MWS), ultimately enhancing student understanding and engagement with mathematical concepts.

A relevant limitation of this study is its focus on a single prospective teacher, which restricts the generalizability of the findings. This case-specific analysis may not capture the broader spectrum of teaching practices and challenges. Future research should include a larger sample to provide more comprehensive insights into the interplay between MTSK and MWS in teaching number sense.

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