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Primary school teachers and their engagement with critical mathematics education via landscapes of investigation

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Abstract
Critical mathematics education is concerned with the connection between mathematics education and social issues. Working with landscapes of investigation is a way to address this concern in mathematics classes. In this article, we seek to understand how primary teachers use landscapes of investigation in their teaching practice from the involvement in in-service teacher education. Specifically, we discuss how Grade 3 and 4 teachers planned and developed landscapes of investigation with their students. This study follows the interpretative paradigm with a qualitative approach. We use case study methodology and participant observation with different integrated data production instruments, which include video and audio recording of the teacher education sessions, field notes and interviews. For the analysis, we use categorical content analysis. The results suggest four characteristics that stood out during the teachers planning and development of landscapes of investigation, related to their connection with the problematization of a social issue, the use of a semi-reality to explore this problem, the format of the tasks and their potential to stimulate critical reflections.

Keywords: primary school, teacher education, professional development, landscapes of investigation, critical mathematics education

INTRODUCTION

In mathematics education research, studies related to professional development of primary school teachers have addressed a wide repertoire of themes (Bakirci & Karisan, 2018; Julio & Silva, 2018; Nacarato et al., 2009). Some of these studies have researched the implications for teachers' didactic knowledge from their involvement in specific teacher education programs. Such implications relate, for example, to the way they plan and enact mathematical tasks (Afini & Silva, 2024; Biccard & Wessels, 2017; Ekawati & Hohar, 2016; Pinheiro et al., 2019), to the type of communication established in the mathematics class (Brown, 2017; Putri et al., 2015), teachers' knowledge of mathematical content (Vasconcelos et al., 2021), and teachers' collaboration practices (Misfeldt & Zacho, 2015; Richit & Ponte, 2019; Quaresma & Ponte, 2019).

An aspect still underexplored in these studies is the fact that many primary school teachers have limited opportunities during their academic trajectories to engage in experiences that involve investigative processes, mathematical discoveries, and the exploration, through mathematics, of social issues such as social inequalities, climate and environmental changes, racism, sexism, violence and intolerance (Gutstein, 2018). The literature has pointed out the importance of such practices for mathematical learning and for the development of students' critical competencies (Gutiérrez, 2013; Gutstein, 2016; Skovsmose, 2023). Additionally, official curriculum documents emphasize the need for school mathematics to incorporate investigative aspects and contribute to understanding social issues. However, many primary school teachers do not feel adequately prepared to meet this requirement, which leads them to seek professional development in this field (Afini & Silva, 2024; Esmonde & Caswell, 2010; Wright et al., 2023).

Although this situation is known, there are still few studies specifically aimed at understanding the way in

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Contribution to literature

- Going beyond previous studies that explored different aspects of primary school teachers' mathematics education professional development, this study strives to understand these aspects from the perspective of critical mathematics education.
- This study responds to this existing research gap by examining, based on a professional development course, how primary school teachers employ the educational approach of landscapes of investigation to foster a critical stance in their mathematics teaching practices.
- The findings show that this approach involves connecting mathematics to socially relevant and relatable issues for children, utilizing semi-reality tasks to explore different topics, empowering students, and encouraging critical reflections.

which primary school teachers learn and lead investigative tasks that allow the treatment of social issues because of their engagement in professional development processes. The research discussed in this paper is a response to filling this gap. Its aim is to understand, from the perspective of critical mathematics education, how primary school teachers involved in in-service teacher education courses use the work with landscapes of investigation in their teaching practice with a focus on the exploration of sociopolitical issues. Specifically, we discuss how Grade 3 and 4 teachers planned and developed a landscape of investigation with their students. We seek to answer the following question: how do primary school teachers put into action the work with landscapes of investigation during the planning and enactment of tasks with their students, based on in-service teacher education?

Critical Mathematics Education

Critical mathematics education can be understood as an expression of concerns regarding mathematics education and the different sociopolitical roles it can play in society (Skovsmose, 2023). The concerns involve what happens in the mathematics classroom and the opportunities in the future life of students; socioeconomic exploitation, racism, sexism, homophobia, xenophobia and violence; mathematics itself, not only school mathematics, but that used in different contexts and in different professions and that directly interferes in people's lives; democracy and social justice; the connection of mathematics education with issues such as equity, inclusion, power relations, foreground of students and teachers, and dialogue (Skovsmose, 2023).

Different versions of critical mathematics education have emerged in recent decades, based on varied theoretical frameworks and traditions. Even with their distinct aspects, these versions[1] share underlying concerns and characteristics: they are driven by urgent and complex issues; they are interdisciplinary; they are politically active, engaged, and democratic; they involve critique; and they are reflective and self-aware (Andersson & Barwell, 2021). These characteristics favor the development of educational approaches that go beyond mere number manipulation and solving application problems by providing students with opportunities to question the very notion of correctness and absolute rationality of mathematics (Skovsmose, 2023) while providing them with mathematical skills to understand the political structure of inequalities, the practices of social injustice, the mechanisms of oppression and to act to transform the world (Gutiérrez, 2013; Gutstein, 2016; Skovsmose, 2023; Stinson et al., 2012). One of these educational approaches is to work with landscapes of investigation. This approach is concerned with developing critical competence in the students by providing the exploration and development of meaningful activities that provide the construction of mathematical concepts and ideas (Penteado & Skovsmose, 2022; Skovsmose, 2023).

Landscapes of investigation are learning environments alternative to the paradigm of exercises, in which students work essentially with repetitive exercises, usually with one and only one correct solution and with application exercises, based on similar examples presented by the teacher. In landscapes of investigation, students are invited to engage in tasks that provide opportunities for them to:

(a) engage in exploratory and investigative processes, which include actions such as creating lines of reasoning, conjecturing, testing hypotheses, conducting tests, raising and defending ideas, constructing new hypotheses or creating models (Ponte et al., 2016; Skovsmose, 2023) and

(b) explore and critically reflect on socio-political issues, many of them controversial (Skovsmose, 2023).

[1] There are a variety of approaches in mathematics education that share the concerns of critical mathematics education. For example, mathematics education for social justice (Bartell, 2018; Gutiérrez, 2009; Gutstein, 2016; Frankenstein, 1983), project-based mathematics education (Vithal et al., 1995); culturally responsible mathematics education (Greer et al., 2009); and ethnomathematics (Powell & Frankenstein, 1997; Knijnik, 2007).

The communicative pattern in a landscape of investigation is based on dialogue, understood as a type of communication that emerges in educational processes (Milani, 2020; Skovsmose, 2023). Dialogue is related to the search for knowing something and questioning, aiming at learning. It is an open process, since its course is unpredictable. It also incorporates elements of equity, as what matters is the content that is said and not the participant's position (Skovsmose, 2023).

Teacher Education Cycles in Critical Mathematics Education

Teaching knowledge is plural and heterogeneous, and teachers develop professionally from reflecting and acting in their practice (Ponte, 2012a). To understand how teaching knowledge is constituted, Ponte (2012b) suggests the existence of four dimensions: knowledge of mathematics to teach, of the curriculum, of students and of teaching practice. This last dimension has a central role, articulating with the others. Its essence concerns the knowledge necessary to lead students' learning situations. The includes several elements, such as the preparation of each lesson and long-term planning, the choice, adaptation and preparation of tasks and the way they will be carried out in the classroom, including the way the students are organized, the creation of a learning culture, the choice and configuration of how communication will take place from the tasks, the way in which the mathematical representations of the contents covered will be made, and the processes of assessment.

If teachers do not have a background that allows them to work from a critical and investigative perspective with social issues, they will tend to reproduce the lessons they had during their basic school education, usually based on the paradigm of exercises (Julio & Silva, 2018). Specific professional teacher education practices, focusing on the development of knowledge of teaching practice related to the concerns of critical mathematics education, can be a way to contribute to overcoming this paradigm. There is extensive literature related to the characteristics that favor the effectiveness of programs aimed at professional development. Some of these studies highlight the need for these programs to offer opportunities for articulating the knowledge of teaching practice (Zehetmeier & Krainer, 2011), to be sustained and planned for the teaching work context (Desimone, 2009); to situate professional learning based on the circumstances of the participants' classes (Loucks-Horsley et al., 2010); to propose collaborative actions that are articulated with elements of teachers' practice (Misfeldt & Zacho, 2015) and offer spaces for reflection based on such experiences and circumstances of the students' learning (Misfeldt & Zacho, 2015; Timperley et al., 2007); and have a long duration or are carried out in a continuous format (Desimone, 2009; Timperley et al., 2007).

Taking these aspects into account, this study developed a teacher education activity aimed at enhancing teaching practices, focusing on working with landscapes of investigation with primary teachers in Portugal. The practical model *Teacher Education Cycles for Critical Mathematics Education* (Afini & Silva, 2024) was used. The model comprises four interconnected stages designed to create opportunities for teachers to reflect and to act on their educational contexts, from the perspective of critical mathematics education.

Stage 1 (E1) covers moments of discussion of ideas related to the work with landscapes of investigation. Teachers are invited to engage in professional learning tasks related to mathematical concepts, semi-reality, and reality. It is a moment in which they experience different landscapes of investigation that can be carried out by their students while reflecting on the possibility of their use or adaptation.

Stage 2 (E2) involves the collaborative planning of one or more landscape of investigation. It is a moment in which teachers return to the experiences of the previous stage, recontextualizing and reinterpreting them. At this stage, the facilitator supports the work of the teachers by offering ideas, suggestions for dynamics and for the format of materials to use. At this stage, teachers "test" the proposals they have developed and often strive to anticipate situations that may arise during the lesson with the class.

The tasks are carried out in Stage 3 (E3). At this point, the teachers pay special attention to how students' learning happens, reflect on the approach adopted in the classroom and identify the challenges and learning opportunities provided by the planned landscape. Teachers are encouraged to report on these topics as soon as the lesson ends, using written narratives (or narrated in audio), and writing of a report or other forms.

In stage 4 (E4), there is a collective reflection on the practice carried out within the formative context. Participants share their experiences, report the dynamics used, highlight the main results and evaluate the areas for improvement, change or continuation. They also discussed their difficulties and what was significant in the lesson, receiving feedback and suggestions. Once the four stages are completed, a new cycle begins.

METHODS AND MATERIALS

Participants and Teacher Education Context

This study is based on an in-service teacher education course for primary teachers with 75 hours (37.5 hours face-to-face meetings and 37.5 hours autonomous work) that used the model of the Teacher Education Cycles for Critical Mathematics Education. The participants were

12 primary teachers from a school group in the Lisbon region (referred to as below with fictitious names). They were from the same school group and nine of them worked in the same school. All but two had more than 20 years of experience. The course was developed in their school context within the scope of the university-school partnership. The teachers were invited through the primary education coordination of the school group, motivated by experiences that provided "external connections" to mathematics, an element indicated in the Portuguese curriculum (Canavarro et al., 2021). Two teacher education cycles were held during the year of 2024, one with 6 sessions (first semester) and the other with 7 sessions (second semester). The facilitator was the first author of this article. The sessions took place after classes, starting at 5 pm and lasting 2 hours 30 minutes. There were also sessions in which the teachers met in teams at alternate times to progress the planning and produce teaching materials, in addition to carrying out planned classes with their students.

Data Production and Analysis

This study follows the interpretative paradigm with a qualitative approach. The case study methodology (Yin, 2018) and the participant observation method (Lüdke & André, 2013) were used with different integrated data production instruments: video and audio recording of the sessions (E1, E2 and E4), notes from the field notebook during the lessons (E3), interviews, the materials used and explored by the teachers during the sessions, the materials produced by them for the lessons and the interactions and teachers interactions in a virtual environment where the course materials were deposited. All data were transcribed. For the analysis, categorical content analysis (Bardin, 2016) was used, with the support of the MAXQDA software. The theoretical perspective used was the critical mathematics education related to working with landscapes of investigation (Skovsmose, 2023) and elements of didactic knowledge related to teaching practice (Ponte, 2012b).

The analysis was carried out in three stages: pre-analysis; exploration of the material; and treatment of the results, inference and interpretation (Bardin, 2016). In the pre-analysis, the data was organized in MAXQDA. The different documents were organized, with floating reading. The aim was to constitute the *corpus* of the research, that is, the part of the data set that would later be submitted to analytical processes. Patterns or initial critical points were searched, choosing and filtering the data for later stages. In the exploration of the material, the *corpus* of the research was explored in a more systematic way. The initial coding of the recording units was established, that is, the excerpts of speeches or dialogues, attitudes assumed by them, situations that occurred in the context of the course and others that were significant. Each code was described, according to its context. The coding process was done through a deep immersion in the data, which culminated in different movements back and forth, of coding and decoding. Subsequently, recurrence and non-recurrence were searched between the codes, and different regroupings were made. This movement provided 12 thematic axes, some of which were built from theoretical elements related to working with landscapes of investigation and others that emerged from the data. In the treatment of the results, inference and interpretation, the data were treated in such a way that to be meaningful. The use of MAXQDA supported this movement, as it allowed the construction of matrices and tables of recurrence between the thematic axes and the manipulation of the codes contained in these themes. Focusing on the research question and using the theoretical framework of the study, several categories of analysis were constructed. In this article, we discuss the results of a thematic axis, which concerns the participants' understanding of the characteristics of landscapes of investigation in primary education. This is done from the analysis of two groups of four teachers (grades 3 and 4 groups) during the first cycle of the course (**Table 1**). We analyze the work done by them in planning and enacting a landscape of investigation with their students.

In Stage 2 (E2) of the course (sessions 4 and 5), the participants were invited to plan a landscape of investigation to be carried out with their classes, in the context of the mathematical content they were working on. The landscape was carried out in Stage 3 (E3) in their own classes.

Grade 4 group

This group was formed by four teachers: Eunice, Fátima, Rosa and Teresa. Teresa and Rosa were teachers and Fátima and Eunice were pedagogical coordinators. They devised a landscape of investigation involving mathematical operations, monetary values, estimates and percentages, and awareness of overspending. They planned three tasks carried out in Teresa and Rosa's classes in six one-hour classes, over three days. In Task 1, in groups, the students were invited to evaluate the budget of a fictitious family, which had expenses greater than its income. The family would save 10% of their income every month for emergencies. There was an excessive expense with clothes and accessories (250€). Initially, the students were invited to perform calculations and to verify if the family's income was sufficient to cover expenses. Finding that the budget was excessive, they were invited to prepare a new budget, choosing items that were essential and non-essential. In Task 2, the children visited the school's "lost and found" and selected samples of different clothes. Using fractions and percentages, students needed to assign fair values to each article, considering its condition and the value of a new piece. They used the internet to do this research. In

Table 1. Dynamics of face-to-face sessions

Stage	Theme	Brief description
E1 Session 1	Investigating areas with tangram and estimating the deforestation of the world. Base text: Skovsmose (2000)	The main ideas about landscapes of investigation were discussed. The teachers experienced a landscape consisting of four tasks. In the first three, they carried out investigations with tangram to work on different notions of area and mathematical relationships between the pieces of the game. In the last task, they used the ideas discussed to estimate deforestation in the world and think of alternatives to face this situation.
E1 Session 2	Pythagorean table, multiplication tables and decision-making in healthy food purchases. (Part 1) Base text: Ponte et al. (2016)	The moments of an investigative lesson were discussed. The teachers experienced a landscape of investigation formed by six tasks. In this session, the first three were worked on, carrying out investigations on the mathematical properties of the Pythagorean table. The teachers explored several properties and worked with different mathematical operations in this process, including the multiplication table.
E1 Session 3	Pythagorean table, multiplication tables, and decision-making in health food purchases (Part 2) Base text: Canavarro (2011)	The role of the teacher in a landscape of investigation was discussed. The teachers used the concepts discussed in the previous session to explore the cost of eating healthily. They experienced a task in which the fictitious Gorgonzola family had an unhealthy eating routine. The landscape of investigation invited them to analyze the food consumed by this family, propose a healthier diet and compare, in brochures from different supermarkets, the cost of a healthy diet.
E2 Sessions 4 and 5	Collaborative planning and production of a landscape of investigation.	The importance of planning in a landscape of investigation was discussed, as well as how to produce investigative tasks and manage the investigation with the students. The teachers were invited to collaboratively plan a landscape that addressed a social issue that was relevant to the context of the mathematical content they were working on with their students. The twelve teachers were divided into three groups, according to the grade in which they worked (grades 2, 3 or 4).
E3	Realization of the planned landscapes of investigation	For three weeks, the groups carried out the planned landscapes of investigation. There was follow-up by the facilitator and individual meetings to support the groups.
E4 Session 6	Collective reflection and evaluation of the work done	The enacted landscapes of investigation were discussed. The teachers evaluated the work, pointed out possibilities for improvement in the tasks, the positive aspects and the difficulties they faced.

Task 3, with the pieces labeled, a "fair" was created in the classroom. Each group had 60€ to make purchases of the items and needed to decide which pieces to buy, to make the best use of the money. One group was responsible for the sale. The children manipulated fictitiously and performed the necessary calculations. At the end, the teachers conducted a whole-class discussion, leading the children to reflect on the excessive spending and on the amount that could be collected for the school with the sale of the clothes of the lost and found. The children made estimates and provided ideas on how to use this resource.

Grade 3 group

This group was formed by four teachers: Angélica, Ema, Maria and Paula. Angélica and Ema were class teachers at the same school and Maria was an assistant teacher in both classes. Paula was also a teacher but in another school. The landscape involved working with a sequence of four tasks and sought to explore and problematize issues related to discrimination and prejudice through mathematics. The tasks were carried out in six one-hour classes, over three days, in the three classes. In Task 1, students were invited to explore and answer an online questionnaire with seven sections. Each section presented the image of different people and information about a social position or some event. The children had to choose the person they believed related to the fact presented. For example, one section had the question: "who could be president of a country?" and just below it the images of a black man, a white man, a black woman, a white woman and an option "anyone". In this example, all the people were presidents of some country. In some questions, the teachers used artificial intelligence to generate the images, and, in others, they used images of real people, taken from the internet. In Task 2, after the three classes answered the questionnaire, the students, in groups, explored and organized the data from the answers to a question, drawn by the teachers. For this organization, they could draw tables or use their computers. In Task 3, each group explored different types of graphs that could be used to expose the data and build the one that best represents the information. They used various materials such as cardboard, colored pens, caps and strings. Finally, in Task 4, each group presented the graphs, explaining how they organized the data and built the graph. At this point, a whole-class discussion was held about the results and about the possible reasons that led the children to pre-judgments in their choices.

RESULTS

The landscapes of investigation elaborated and enacted show four main characteristics:

1) They explored a relevant social problem that was close to the children's reality,
2) They use semi-reality to explore the problem,
3) The tasks present space for the use of different mathematical strategies and obtaining different answers, providing decision-making power to the children, and
4) The tasks favor critical reflections. These characteristics indicate how the teachers developed their understanding of working with landscapes of investigation with primary students, and they are addressed in detail in this section.

Exploration of a Socially Relevant Problem Close to the Reality of the Students

During the first session focused on planning (E2, session 4), the teachers discussed aspects of learning related to the social issue and the mathematical content to be addressed in the tasks composing the landscape of investigation they were planning. They began with a brainstorm about different social issues that might be suitable for working with children aged 8 to 9 years old. During this process, their teaching experiences were compared with those from the Stage E1 sessions of the course. Below are some moments of the discussion where this becomes apparent:

Angélica: OK, social issues. We have several. Violence, respect for others.

Ema: But how do you put this into practice in mathematics?

Angélica: Look, the number of times they call, they swear at each other...

Paula: There's more. What else do we have? We have....

Ema: For me it's the question of the value of things, of social issues, of being able to buy.... No, no... For me, it would be the theme of healthy eating. Because I've never worked on these issues there, I think it's complex.

Angélica: We begin with this, and we can go to democracy, which is the other...

Emma: But how do you do that? In terms of working with them.

Angélica: Statistics, data processing...

Ema: But how are you going to get that data? On what basis?

Angelica: With questionnaires.

Paula: It depends on what we want to work on in the part of democracy... We can use an online form.

(E2, Session 4, Grade 3 group)

Rosa: For example, healthy eating...

Teresa: How?

Rosa: Within food, you create a problem. They invent a problem that they solve with the additional algorithm and all the data from the image. "Write and solve your problem" ...

Teresa: But this one is very easy. However, you're going to buy this, you're going to buy that, you're going to buy that other one. What are they going to investigate?

Fatima: So, it's just addition, the paradigm of exercise.

(E2, Session 4, Grade 4 group)

In the first excerpt, the Grade 3 teachers discussed various themes that could be relevant for working with their classes, negotiating which social issues could be addressed. In the second excerpt, Rosa suggested that the tasks should encourage students to elaborate on problems related to healthy eating and solve them using the information provided. Both groups aimed to articulate a social problem appropriate for the ages of the students they worked with. After some deliberation, the Grade 3 group selected discrimination and prejudice as their social issues, while the Grade 4 group chose excessive consumption. During the initial planning phase, the teachers were concerned about ensuring that the tasks would be meaningful to the students. The solution they found was to address the social issue through something relevant to the children closely connected to their reality. For example, in the following excerpt from a discussion in the Grade 4 group, the teachers began by examining the importance of raising awareness about excessive consumption. They then shifted their focus to a school-related problem: the large number of forgotten clothes and accessories left unclaimed by parents and scattered on the "lost and found" counter:

Rosa: We could do it like this: each child could look at what they have at home, the extra clothes sitting in their closets. Clothes they haven't worn in, I don't know, forever...

Eunice: Look, no. Let's not take it that far… We can get their attention with what's already here in the school. That pile of forgotten clothes. We could write something, maybe make a poster or something like that...

Fátima: But I don't know how we would even start. Because we need to introduce the mathematics content. We must explain what we're trying to do.

Teresa: We're starting from the social issue of overconsumption. We can work with that, about waste. But first, we need to understand what mathematical activity is. Then, the kids themselves would realize that it's not beneficial for families to keep coming here every day to buy clothes, a piece of clothing.

(E2, Session 5, Grade 4 group)

Once a social issue was defined, the teachers needed to relate it to mathematical learning objectives within the context of their classes. This process of negotiation between the social issues and the mathematical content was marked by back-and-forth movements and took some time to articulate. It was only at the second planning session (**Table 2**), after the social issue was decided, that they defined the mathematical learning objectives to be developed, based on the Portuguese curriculum, and began preparing of the accompanying booklet and planning how the students' mathematical learning would be conducted:

Fátima: Fractions. Because we will say some like of this: the price is 50€. The real price. So, we will sell by half. Or is the fourth better?

Teresa: I think we also could work with percentages. This is there.

Rosa: Estimates, percentages, addition, money operations.

Fátima: We could propose a sample. Write there in the plan. To market the lost-and-found clothing from a small sample. Five pieces of clothing. Because we cannot take everything from there. Five pieces. Five T-shirts, five cold weather sweaters. Five of each one.

(E2, Session 5, Grade 4 group)

Table 2. E2, session 5, grade 3, group lesson plan

Learning to promote	Define what data to collect in a study and where it should be collected. Create and analyze graphic representations and critically discuss their adequacy, developing statistical literacy. Draw conclusions, substantiate decisions and raise new questions raised by the conclusions obtained, to be pursued in possible future studies.
Aims	Observe, select, collect and analyze data. Understand various data representations, to build a graph. Critically reflect on the different "value judgments" and prejudices that exist.

The definition of the social issue and the mathematical learning objectives did not follow a linear pattern. In both groups, the social issue was the starting point for the developing the landscape of investigation. However, the approach to understanding this issue and linking it with the mathematical contents of the curriculum required significant effort in the panning process. In addition to the two sessions of Stage E2, additional meetings were necessary for the teachers of each group to finalize and test the materials that the students would use during class. In other words, determining the "how" to use mathematics to address social problems proved to be a challenging aspect for both groups. Moreover, during the undertaking of the tasks (Stage 3), some tension was observed in balancing mathematical objectives with those related to the social issue. For instance, Paula, during the class, noticed at one point that the students seemed to be disconnected from the social issue while manipulating the data to construct the graphs. In Session 6, she addressed this situation and explained how she responded:

Paula: For me, it was very important, in fact, it was the moment when we took the break, because I think that these things would only have a social part later, if we worked on systematizing at that time. If not, what was going to happen? Things were going to be up in the air, and I thought that would not be correct either for them or for me. Therefore, I told the teacher, I'm going to reverse this, I'm going to give my students 20 minutes to finish, and one of the groups had to redo it, because the graphs weren't adding up. "You have 20 minutes to finish the graphics and it's time to finish to present." And then, for me, it was the richest part, it was that they then used mathematics to create arguments within themselves, to be able to justify what they were doing.

(E6, Session 6, Collective reflection)

Paula chose to "accelerate" the work related to the construction of the graphs so that the systematization of the task could happen and make it possible to carry out the whole-class discussion related to the theme of discrimination and prejudice, within an adequate time. She felt that she needed to reduce the children's time working with mathematics so that the objectives related to the social issue could be achieved.

Using Semi-Reality to Explore a Social Problem

The teachers employed semi-reality tasks to trigger the children's investigative work. This approach likely occurred, in large part, due to the influence of the teacher education course. As previously mentioned, during Session 3 of Stage E1 (see **Table 1**), the teachers experienced the task involving the Gorgonzola family, who had an unhealthy eating routine. They were invited to analyze the foods consumed by this family, think critically, and propose a healthier diet. Later, they used leaflets from different supermarkets to compare the high cost of maintaining a healthy diet. The use of semi-reality to address a social issue – specifically, the high cost of healthy eating – seemingly influenced the teachers' planning when discussing the topic they intended to address with the children. This influence can be observed in the following dialogues:

Eunice: What if we say this using a fictional family? [Takes the task of the Gorgonzola family into her hands]. Look at this one we made: "The Gorgonzola family's eating routine seems unhealthy. Discuss it with your colleagues. Was this family's attitude correct or not, correct? Shall we help the family? To do this we need to prepare a new menu..." [begins to leaf through the task and show it to the other teachers in the group].

Fátima: And the family could spend a lot on clothes.

Teresa: The idea is for the children to understand we shouldn't buy new clothes every month. From this, we make a story about where the mother or father is a habitual spender. At the end of the month, the family doesn't have money to pay for water or gas.

Fátima: And then we have our entrance here, where there are a lot of clothes there. Because kids don't take care of their clothes, because they have so many clothes each one that they don't even know what's missing.

Teresa: And the parents also don't care about the result, nor do they know it.

Rosa: Yeah, we could also talk at the end about what to do with those clothes from the school counter. If no one comes to look for them, what can we do with them? What is your solution for those clothes? Donate? Sell them?

(E2, Session 4, Grade 4 group)

Angélica: Maybe here we could use semi-reality. We could invent a situation so that they could explore the issue of prejudice.

Paula: Yes, and not only in the investigation part.

Ema: It can also be.

Paula: We can also provide this information. We give them some information to explore.

(E2, Session 4, Grade 3 group)

Using semi-reality to introduce a social issue proved to be a way to avoid individualizing situations within the specific context of children. For example, in grade 3, the teachers chose to collect the answers to the questionnaires anonymously. In grade 4, the teachers created a fictitious family to explore their budget, instead of asking each child to inquire about their own family's expenses. This approach brought the problem into the children's context in a softer, more age-appropriate manner. Another aspect related to the use of semi-reality is connected to the use of textbooks. Usually, mathematics textbooks present many tasks based on the paradigm of exercise and offer semi-reality exercises only to apply algorithms or concepts. When beginning to prepare the tasks, one strategy used by the teachers was to take the exercises and problems from the textbooks and adapt them into an investigative format:

Fátima: [Starts reading an assignment from the textbook] "John finished college and started working. He has a monthly income of 1200€. So that the money can cover the monthly expenses, he made a budget. Look, he has these expenses: car payments, help with his parents' household expenses... After paying all the expenses, what balance does João have?"

Rosa: But this is the paradigm of exercise format.

Fatima: OK. But then we begin with that.

Teresa: Yes.

Fátima: And now, for example, how do they investigate?

Teresa: We put in three or four things that aren't important, to see what they take out. For example, they could withdraw the amount spent on clothes. The others could have other things. Another group might take it, I don't know, from the food. And then we will find what is superfluous for some and not for others.

Eunice: And then, that could be the idea, it could be "you need to justify the choice".

Fátima: That's right. Of course, of course.

Teresa: What could it be, if they came to present the budget, would they give the type of accounts...

Rosa: Yeah, because if you take your clothes off, how much money will it be better? Will there be any balance?

Fátima: We give a budget, and they create a new one.

(E2, Session 4, Grade 4 group)

Tasks That Allow the Use of Different Strategies and Responses and Provide Decision-Making Power to Students

In the three sessions of Stage E1, the teachers discussed characteristics of the landscapes of investigation and the different moments of a lesson with this approach, including the teacher's role during the student's investigative work. They also experienced different tasks that involved investigating mathematical properties and their connection to social issues (see **Table 1**). The analysis indicates that some of the characteristics discussed during these sessions remained with the teachers as they planned and enacted the landscape of investigation in their class. In general, the landscapes:

1) Provided students with opportunities to use different solution strategies,
2) Presented questions that had different answers, and
3) Empowered students with choices.

For instance, in Task 1 of the Grade 4 group, children were invited to evaluate a family's monthly budget, which included excessive spending on clothes and accessories. While carrying out the task, they performed calculations (in different ways) and realized that the family's income was insufficient to cover the budget. They were then invited to prepare a new budget that fits within the family's income, which led to different responses among the students. During this process, the children had the power to choose the items to remove, add, or even modify the values. This generated and enriched the whole-class discussion:

Rosa: Many of the kids said that there was no point in spending 250€ on clothes. They thought it was a lot of money; it was excessive and that it was not justified in spending that money every month. Therefore, they immediately excluded 250. They also excluded... There was a boy who saw and said, "look teacher, but also, meals in restaurants, 60€, three people, also only go once". Oh, that was cool. Then again, it doesn't happen that often. I said, "Yeah, but maybe, what do you think? What do you think?" They said, "Oh no, then let's take it off. If the money isn't enough, we won't go to the restaurant."

(E4, Session 6, Grade 4)

In Task 3, which involved using fractions and percentages, the children also had the power of choice to select the pieces of clothing from the "lost and found" and to assign a value they considered fair. This value was based on the condition of the clothing and the price of a new item, determined through an online search. For example, Teresa highlighted how she found it interesting to encourage the children's decision-making skills and to listen to their reasoning and justifications:

Facilitator: In this case, you standardized, for example, look, the used ones will be half of the value, a third of the values, or are they the ones who decided and you just helped?

Teresa: No, they decided. Also considering the state of the piece in question, because there were pieces that were practically new and then they said, "teacher, we can't reduce the value by half, it's not fair". We speak, in fractions, a third, a quarter. Moreover, I said "Oh no, maybe not", because there were, for example, sweatpants that even had patches on the knees. In addition, they thought, "no teacher, half no, it's not fair, the piece is no longer in good condition", others said "it's really for those neediest families who really need that article". And that was very funny, it was very funny that they gave the values and the arguments they used.

(E4, Session 6, Grade 4)

Similarly, the grade 3 teachers expected the students to independently choose different ways to explore the data they were working with and, based on that, decide which types of graphs to use. During the planned lesson stage, no specific path was outlined for the students to follow during the treatment of the data, leaving them free to make their choices while working on the task.

In both areas of investigation, the teachers did not anticipate all the possible situations might arise, despite their attempts to do so during the planning stage. In the case of grade 3, the teachers expected some responses to reveal preconceived judgments about the situations presented to the children, based on the images accompanying each question. This element, in their planning, would guide the whole-class discussion toward the social issue addressed in the landscape of investigation. In the grade 4 tasks, except for Part 1 of Task 1, there was no single solution. The students could either explore or not explore the issue of the excessive budget allocation for clothing and accessories. This aspect of the task was also crucial for sparking discussions about overconsumption and the exploitation of an excess of clothing left forgotten at school:

Rosa: We were imagining the students' responses and couldn't think of everything.

Teresa: Oh, they're going to surprise us... They're going to surprise us...

Fátima: That's what investigation is... As much as we... I'm assuming they're going to say this, I'm assuming they're going to say that... As much as we try to anticipate what might happen, we never anticipate everything.

(E2, Session 5, Grade 4 group)

Thus, the teachers understood that it would be necessary to take some risks in carrying out the tasks. In other words, it was not possible to accurately predict the path that the children's choices could take. This was not seen as an obstacle, but as the potential of the task of achieving mathematical and social aims.

Tasks that Stimulate Critical Reflections

The way in which the teachers understood the work with landscapes of investigation was by creating opportunities to stimulate the students to reflect critically. In the groups analyzed, these reflections occurred in different ways. In the grade 3 group, it happened during the whole-class discussion of Task 4. The children presented their graphs, explained the way they had created them, and discussed the reasons that might have led their classmates to choose a certain answer. The teachers stimulated a dialogue to problematize these answers and foster the children's reflection. For example, in the question "Who could be the president of the republic?", more than 80% of the children (across the three classes) chose the image of a white man. There was no choice for the two women depicted in the images. When one of the groups presented their graph, Paula led a dialogue to encourage the children's critical reflection on the situation:

Paula: Why did the classes imagine that the president of the republic would be this man?

Student 1: Because there are more men running for president.

Student 2: Because this man had more of a president's look.

Student 3: Because he's sitting in an office.

Student 4: Because he dresses like a president.

Paula: And why did no one choose the woman as a possible president?

Student 5: Because in the photo this woman is inside the house.

Student 6: This woman doesn't look like a president.

Student 7: Because she's too young.

Paula: So, no woman can be president? Should only men be? What do you think? So, a president can't show up from his home, he always must be in the office?

Student 5: Anyone can be.

Student 6: Maybe I wanted to know the name of the president.

Paula: [Opens the computer and shows a news article] The woman that no child in the three classes chose was the president of Mexico. All the options were from people who were presidents. The children enjoyed seeing a female president and there was a lot of talk about it.

(E3, Researcher's field notes, Grade 3)

In the same way, the landscape of investigation elaborated by the Grade 4 group made a movement to offer situations for children's critical reflection, in different ways:

Rosa: The most interesting thing was that all the groups mentioned that the amount

> presented in the budget for "clothing, footwear and accessories" (250€) was very exaggerated. Two of them withdrew from the new budget, considering it to be unnecessary spending. Some children suggested reusing the clothes they already must save money.
>
> (E4, Session 6, Grade 4 group)

> Teresa: Another thing that made a big impact was that with 10 pieces of clothing they raised 112€. So, I said, "OK, now imagine if this were true and if we sold everything that's down there in the box of the lost and found. Oh, so much money!" There was a kid who said, "teacher, could you help with the rental of buses for study visits".
>
> (E4, Session 6, Grade 4 group)

In the excerpts highlighted, the teachers report that the students critically reflected on the problems worked on in the landscapes of investigation, using mathematics to do so. Paula used dialogue to problematize the importance of not having pre-judgments in positions occupied by women. Rosa considered it important that all students understood that 250€ a month spent on clothes, for a small family, seemed excessive. Teresa also considered it critical that the children perceived that the volume of clothes forgotten at school was too much and that, if they were sold, it could provide benefit to the school.

DISCUSSION

One of the main aims of the course conducted in this study was to offer opportunities to expand the participants' knowledge of teaching practice related to working with landscapes of investigation. These opportunities are transversal in the teacher education model used and involve experiences with landscapes of investigation, engagement in the collaborative, development of tasks and their enactment, with room for reflection and discussion with other teachers. By closely examining the ways in which the grades 3 and 4 teachers implemented work with landscapes of investigation, it is possible to verify a movement of overcoming the paradigm of exercise. The results discussed here highlight four key characteristics that emerged in this process. The first characteristic highlights the connection between landscapes of investigation and the problematization of a social issue is highlighted. This is important for practice from the perspective of critical mathematics education (Gutstein, 2016; Skovsmose, 2023). Landscapes of investigation expose students to a wide spectrum of problems, which include mathematics and its applications, but also social issues of different nature (Skovsmose, 2023). During the planning stage (E2), the teacher's choice related to the nature of social issues was designed to be both relevant and appropriate for children, aged 8 to 9 years old. To achieve this, they used problems that were close to the students' reality (Miranda, 2015; Gutstein, 2016). In other words, they chose to explore a social issue that children could recognize and engage with. This was the common thread guiding the planning of the tasks. Many teachers have social and political intuitions and, in some way, perceive the existence of an interaction between mathematics and these intuitions, but they do not always know how to implement them in ways that help their students understand these connections (Gutiérrez, 2013; Xenofontos, 2019). In the case of primary school teachers, who teach all subjects, identifying the social issue seems to involve fewer difficulties. However, aligning it with mathematical objectives can be challenging. The teachers spent a significant amount of time during planning to align the mathematical content with the social issues defined. Moreover, during the lesson, it was necessary to find a balance between mathematical and social learning. Paula, for example, chose to reduce the time of mathematical exploration while the children carried out autonomous work to create the graphs. This allowed more time for discussion of the social aspect, which was more extensive than originally planned. Negotiating and balancing the objectives of mathematical learning and the social learning of students has been a dilemma for teachers working within this perspective (Bartell, 2013; Brantlinger, 2013; Miranda, 2015). There is no well-demarcated path for achieving this balance, even when each stage of the tasks is planned.

The second characteristic related to how the teachers put into action the work with landscapes of investigation concerns the use of tasks in the semi-reality to explore a social problem. Both groups used fictitious situations to address the subjects addressed. This choice may have been motivated by the experience in Stage E1 of the teacher education course, but also by the sensitivity of the teachers related to the non-individualization of the situations regarding the children. In addition, to design these fictitious situations, they used the textbook as a starting point, adapting and transforming an exercise into a task with investigative potential. In this action, they "opened an exercise" (Milani, 2020), looking for possibilities to adapt it to an investigative task. Creating such openings is important to change mathematics education captured by the paradigm of exercise (Skovsmose, 2023).

Another characteristic concerns the way the tasks were prepared. These tasks offered opportunities for children to use different solution strategies, presented questions with different answers, and explored the children's power of choice. These elements make

investigative tasks very rich for students' mathematical learning (Ponte et al., 2016). The students' power of choice contributes to transforming the nature of mathematics classes. Having the power of choice involves assuming social responsibilities, and this is a concern of critical mathematics education (Skovsmose, 2023). Furthermore, using landscapes of investigation requires an attitude of openness from the teacher. Sometimes, it is necessary to leave the comfort zone to enter a risk zone where it is impossible to control all future events resulting from the students' actions and choices (Silva & Penteado, 2013). In the paradigm of exercises, teachers have almost total control over these actions. In landscapes of investigation, this is not the case, and teachers need to take risks (Penteado & Skovsmose, 2022), as students' questions and the teacher's responses cannot always be pre-formulated. In other words, landscapes of investigation bring changes to teaching practices by including new risk structures into the teacher's work, while simultaneously offering new educational possibilities (Skovsmose, 2023).

A final characteristic related to the teachers' work with landscapes of investigation is the potential of the tasks to stimulate critical reflections. This is part of the investigative work. For example, in a purely mathematical task, critical reflections may emerge when students choose one solving strategy over another or when they reach a generalization through task exploration (Ponte et al., 2016). Similarly, critical reflections may arise when students analyze a situation of social injustice through a task set in a context of reality or semi-reality (Gutstein, 2016; Skovsmose, 2023). In the landscape of investigation of grade 3, the students reflected critically when they realized that the choices related to the roles people could play carried elements of discrimination and prejudice. In the grade 4 group, critical reflections emerged as students used mathematics to understand the problem of the large number of forgotten clothes at school and the issue of overconsumption. The tasks planned by the teachers captured the idea of how mathematics can be an important tool for understanding social aspects and fostering critical reflections on a path to social change (Gutiérrez, 2009; Gutstein, 2018; Skovsmose, 2023). In addition, the landscapes of investigation showed potential to engage children with controversial subjects (Skovsmose, 2023). By developing landscapes of investigation that encourage children to critically reflect on social issues, teachers opened opportunities on topics not traditionally addressed in mathematics classes. For instance, in addition to addressing the issue of overconsumption in grade 4, the landscape of investigation could lead to discussions about the production of clothing that exploits child labor, fair family income, and the environmental impact of overconsumption. Similarly, with grade 3 students, discussions could include exploring why most presidents of countries in America and Europe are white men.

CONCLUSION

This study shows that an in-service teacher education course with primary school teachers, with strong connection between theory and practice (Ponte, 2016) was able to interest teachers in planning and enacting landscapes of investigation (Skovsmose, 2023) in their classes. In doing so, they faced a tension regarding the balance to explore the social issues and the mathematics contents. The way the teachers put into practice landscapes of investigation highlights four main characteristics: the connection with a social issue, using semi-reality tasks to explore a social issue, bringing more power to students' explorations and responses, and providing more power to explore critical reflections. These characteristics clearly enrich the students' learning experiences in mathematics.

In addition, understanding these four characteristics favors the creation of professional learning tasks aligned with the model used in this study, as well as motivating alternative practices for professional development from a critical mathematics education perspective. In this study, the experiences of teachers with such tasks during Stage E1 of the teacher education course proved to be a key element in fostering openness and acceptance among the participants. This not only encouraged their engagement in the landscapes of investigation but also inspired them to envision possibilities for creating and enacting tasks from this perspective in their teaching practice. A pathway to professional development towards critical mathematics education involves positive experiences with professional learning tasks that enable such openness.

Moreover, understanding these characteristics contributes to recognizing the different ways in which landscapes of investigation can connect to sociopolitical issues relevant to primary school students and how these issues may be integrated into classroom mathematics tasks. Through the process of experiencing, discussing, planning, and enacting landscapes of investigation focused on addressing a social issue that is not usually explored in primary mathematics classes, teachers engaged in a movement towards critical mathematics education, realizing that young students can and should use mathematics to understand and get involved with social aspects, and even controversies. This can be a potential way to promote positive changes in mathematics classes, moving beyond the paradigm of exercise.

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