MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2025, 21(7), em2663
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/16565

Physmatic difficulties and students' thinking approaches

Hadas Levi [1,2*], Avraham Merzel [1], Yaron Lehavi [3,4]

[1] The Seymour Fox School of Education, The Hebrew University of Jerusalem, Jerusalem, ISRAEL
[2] Department of Applied Mathematics, Jerusalem College of Technology, Jerusalem, ISRAEL
[3] Department of Science Teaching, Weizmann Institute of Science, Jerusalem, ISRAEL
[4] The David Yellin Academic College of Education, Jerusalem, ISRAEL

Received 06 November 2024 ▪ Accepted 11 May 2025

Abstract

Students often face significant challenges related to the interplay between physics and mathematics, referred to as "physmatics." This invited diagnosing the possible sources of students' difficulties in this respect. The current study aims to identify and interpret students' difficulties in three specific physmatic areas required for physics learning: arithmetic operations, overgeneralization of the mathematical zero-product property, and the region of applicability of a formula, distinguishing these from independent difficulties in mathematics or physics. The research is designed to investigate whether these difficulties originate from pure mathematics limited skills, from applying mathematical thinking to real-world problems, or from applying it specifically to problems in physics. A questionnaire was administered to 199 high school students to examine the influence of the context–purely mathematical, physics, and economic–on their responses. Findings indicate that limited mathematical skills do not explain students' difficulties in solving real-world problems. We infer that the students harness three different content dependent mathematical thinking approaches when addressing the challenges posed to them: context-oriented thinking, mathematical thinking, and unripe-mathematical thinking. Each of these approaches is characterized by varying attentiveness to context. Understanding these thinking approaches can help educators develop instructional strategies that address students' cognitive challenges, potentially improving physics education outcomes.

Keywords: physmatic difficulties, physics-mathematics interplay, real-world context, high school physics education

INTRODUCTION

Difficulties regarding the interplay between physics and mathematics ("physmatics"–a term adopted from Zaslow (2005)) can be defined as difficulties in harnessing and integrating knowledge from both disciplines to make sense of physics equations and problem solving. These physmatic difficulties are described in the literature in several ways: math and physics as different languages (Redish & Kuo, 2015), general errors using math in physics with unproductive framing (Modir et al., 2019), and misalignment of students' cross-use of mathematical and physical objects and tools in reasoning (Gifford & Finkelstein, 2020). For example, the difference between the use of variables and parameters in the two disciplines was recognized as the reason for students' difficulties with using a linear equation to make sense of a phenomenon in physics such as spring compression (Heck & van Buuren, 2019).

A recent summary of empirical studies of students' difficulties (admitted to be non-exhausting) pointed out that examination of younger students is scarcer and that in general they tend to focus on finding patterns among the problem solving strategies and interpret them. Several studies show that students' physmatics difficulties arise mostly from unawareness of the

This article was conducted as part of the PhD study of the first author.

✉ hnlevi@gmail.com **(*Correspondence)** ✉ avraham.merzel@mail.huji.ac.il ✉ yarlehavi@gmail.com

Contribution to the literature

- The study identifies three distinct thinking approaches students demonstrate when facing real-world physics problems: mathematical, context-oriented, and unripe mathematical thinking.
- The study shows that mathematical proficiency alone does not ensure successful transfer to physics contexts, especially in tasks requiring structural skills such as interpretation and validation.
- The study expands the framework of physmatic pedagogical content knowledge (PCK) by linking student thinking approaches to specific categories of physmatic difficulties.

structural role of mathematics, not from deficiencies in its technical role (Pospiech, 2019). In this respect it should be added that students with high mathematical skills tend to have strong extra-mathematical knowledge, which is crucial for solving complex problems by understanding real-world context (Sigus & Mädamürk, 2024).

Based on a case study researchers concluded that it is difficult to discriminate between two levels: using equations more algorithmically and connecting manipulation of equations to their physical interpretation (Hull et al., 2013). Hence, the need to study students' performance at different levels of physmatics.

In previous work (Levi et al., 2020; 2021), we found, among others, difficulties in these two levels: inadequate use of arithmetic operations in physics equations and providing flawed physical meaning to a zero value of a product of parameters and variables. We also identified another difficulty: unawareness to the region of applicability of physics equations. This can be regarded as a third level of physmatics difficulties related to how mathematics is interwoven in the structure of physics as a whole discipline. Such difficulty was reported regarding teachers' awareness of the borders of validity of Galileo's law of free fall (Lehavi & Galili, 2009) and to the complexity of constructing and validating a mathematical model for physical phenomena (Uhden et al., 2012).

A great challenge in diagnosing physmatic difficulties is distinguishing them from mutually independent difficulties in mathematics or physics. Some studies addressed this challenge by comparing students' problem-solving performance in purely physics contexts (PCs) and purely mathematical contexts (for example, Carli et al., 2020; Macuca et al., 2022). These studies, conducted in middle school and higher education levels, found that although there is some correlation between students' performance, they do not necessarily use the same problem-solving strategy in both contexts. Apparently, there is no empirical evidence whether students' difficulties in specific physmatic domains arise from difficulties in performing pure mathematical manipulations, from performing them in a PC or in a real-world context other than physics. In our study we examined in juxtaposition these three aspects in 9^{th} to 12^{th} grade physics classes. Continuing our previous research mentioned above, we investigated the students' responses to three types of problems, each challenges different physmatic comprehension: use of arithmetic operations, and awareness to the ramifications of mathematical zero-product and to the region of applicability of a formula. These problems were situated in three different contexts–purely mathematics, physics and non-physics (economics). Based on our findings, we inferred on the related cognitive difficulties students experienced. Our study may contribute to understanding students' cognitive difficulties in physmatics and by this help to broaden the theoretical framework of teachers' pedagogical content knowledge (PCK) in this respect (cf. Pospiech et al., 2015; Lehavi et al., 2017). This, in turn, may help teachers develop tools to address efficiently such difficulties.

THEORETICAL BACKGROUND

The Role of Mathematics within Science Learning

The sci-math sensemaking framework (Zhao & Schuchardt, 2021) categorizes sensemaking of mathematical equations in science into two dimensions: science sensemaking and mathematics sensemaking. In science sensemaking, categories range from understanding entity labels to identifying variable associations in phenomena. In mathematics sensemaking, categories span from procedural knowledge to deep conceptual understanding. In higher education, the choice of science majors is positively influenced by mathematical and scientific abilities acquired during high school (Zhao & Perez-Felkner, 2022).

In the special case of physics and mathematics, their deep intertwined history within the realm of modern science plays a fundamental role. Indeed, many distinguished figures in history, from Archimedes to Newton and Gauss made a fundamental contribution to the development of both disciplines. The discussions on the relations and distinctions between physics and mathematics go back to Aristotle (Gingras, 2001; Zaslow, 2005). In the context of physics education, it was claimed that mathematical modeling of the physical world should be the central theme of physics instruction (Hestenes, 1987). Hestenes (1987) perceived mathematical models in physics as conceptual representations of real phenomena, with physical properties represented by quantitative variables within the models. According to Hestenes (1987), such a model has four components:

(1) a set of names for the object and agents that interact with it,

(2) a set of descriptive variables representing properties of the object,

(3) equations of the model, describing its structure and time evolution, and

(4) an interpretation relating the descriptive variables to properties of some object which the model represents.

The skills related to components 3-4 were addressed by Pietrocola (2008) who suggested differentiating between technical skills, related to the practical ability to apply mathematical laws and algorithms, and structural skills related to the use of mathematics to grasp the fundamental principles of physical thinking and to discern the profound connection between the context of physics and its mathematical representation. Based on his work, Uhden et al. (2012) proposed to regard the integration of mathematical modeling in physics education as being comprised of three fundamental aspects: a pragmatic facet, used as a tool, a communicative dimension, used as both a language and a cognitive tool, and a structural-functional element, providing the groundwork for logical and deductive thinking.

Regarding physics and mathematics as “intertwining” becomes particularly apparent when dealing with intricate physics concepts, such as velocity and acceleration that require mathematics to gain their full physical meaning (Pospiech et al., 2015).

However, the need to integrate mathematics and physics in the context of physics education faces the fact that they “speak different languages” and represent different cultures: “Math in science (and particularly math in physics) is not the same as doing math. It has a different purpose–representing meaning about physical systems rather than expressing abstract relationships. It even has a distinct semiotics–the way meaning is put into symbols–from pure mathematics” (Redish & Kuo, 2015, p. 563).

In the context of physics teaching, Lehavi et al. (2015, 2017) studied via interviews how physics teachers intertwine physics and mathematics. They observed that the teachers’ strategies for introducing the phys-math interplay can be described by different patterns representing “treks” between the two domains and within each of them. Each of these patterns serves a different teaching goal but they all begin with a physical description of a phenomenon, continue with mathematical manipulations and end in seeking new physical insights. The researchers concluded that the teachers developed a special PCK categorized into five dimensions: orientation to teaching, knowledge of the scientific curriculum, knowledge of students’ understanding of science that reflects in students’ difficulties, knowledge of evaluation, and knowledge regarding teaching strategies. In all of these dimensions both the content and the pedagogical knowledge do not fall strictly into physics or mathematics but rather into their interplay.

The literature indicates the following skills in this respect:

(A) Simplification–the process of transforming a physical phenomenon into a more straightforward form, facilitating modeling (Uhden et al., 2012). This process involves phase 1-phase 2 in Hestenes (1987) framework: choosing (and naming) the minimal number of objects, interactions and descriptive variables required for modeling the behavior of a phenomenon.

(B) Mathematization–the translation of physical phenomena into mathematical formalism (Karam et al., 2010).

(C) Manipulation–a series of mathematical operations that can be categorized into technical and structural (Karam et al., 2010; Pietrocola, 2008; Uhden et al., 2012). Skills in category B and category C may be related to phase 3 in Hestenes (1987) model.

(D) Interpretation–providing physical meaning to the results obtained by making manipulations. It encompasses the ability to decipher equations, extract their meaning, identify boundary cases, generate physical predictions from mathematical formalism, and seeking for analogies. Interpretation is highly regarded by researchers (Redish & Kuo, 2015).

(E) Validation–the concluding step in the modeling process, where the alignment between the outcome model and observation is assessed. It ensures the robustness of the conclusions drawn from the analysis and involves rigorous testing to verify the integrity of the findings. Both interpretation and validation can be connected to phase 4 in the model proposed by Hestenes (1987).

Table 1 summarizes Hestenes’ (1987) modelling framework and the corresponding required skills.

Challenges in Researching the Role of Context in Physics Education

Although all problems in physics are contextualized, they can vary with regard to the level of complexity by which they are presented (e.g., Yerushalmi & Magen, 2006). This can be related to a tendency in teaching to conduct the simplification phase before presenting a problem to the students (Taber, 2000). Consequently, students that begin the modelling process from phase 3 in Hestenes’ (1987) framework may develop partial

Table 1. Hestenes' (1987) modeling framework and its required skills

Hestenes's modelling framework	Required skills
(1) A set of names for the object and agents that interact with it	Simplification
(2) A set of descriptive variables representing properties of the object	Simplification & mathematization
(3) Equations of the model, describing its structure and time evolution	Mathematization & manipulation
(4) An interpretation relating the descriptive variables to properties of some object which the model represents	Interpretation & validation

cognitive tools for addressing the phys-math interplay in physics. Thus, in addition to studying the teaching of the phys-math interplay or the skills students harness to develop a valid mathematical model of a physical phenomenon, there is a need to study students' cognitive challenges in reconciling the possible tension between their mathematical thinking and its use in a real-world context.

There are educational frameworks that address mathematics instruction by contextualizing it in real-world situations, such as realistic mathematics education (e.g., Fraihat et al., 2022). Although there exists evidence that such teaching can enhance students' mathematical learning (Jones, 2015; Reyes et al., 2019), it is very common that students who learn physics construct their mathematical thinking separately by different teachers in pure mathematics lessons and in physics lessons where they develop their in-context thinking approach. Research has shown that students with high mathematical skills tend to have strong extra-mathematical knowledge, which is crucial for solving complex problems by understanding real-world context (Sigus & Mädamürk, 2024). It is important to note that physmatics, unlike the RME, focuses on examining the role of the interplay of mathematics and physics in teaching and learning physics, not mathematics.

One of the challenges in diagnosing physmatic difficulties is distinguishing them from difficulties in mathematics or physics. Addressing this challenge regarding arithmetic difficulties by comparing students' performance in solving 'PC' problems and 'purely mathematical context' problems (Carli et al., 2020; Macuca et al., 2022) revealed that adding a PC increased the difficulty for students beyond performing arithmetic operations in a purely mathematical context. Thus, students may harness two different thinking approaches when addressing a problem: pure mathematics thinking approach and in-context mathematics thinking as two thinking approaches.

Our study aims at finding and characterizing the differences between these two thinking approaches used by students when harnessing certain skills in modeling natural phenomena. In order to find whether adding a PC has a special impact on students difficulties compared to adding other real-world contexts, we added the context of economics to our investigation leaving the required mathematical skills the same.

Physmatic Difficulties

There are several models that describe students' physmatic difficulties.

One model suggested mapping students' physmatic difficulties onto a two-dimensional plane spanned by two perpendicular axes: physics-mathematics axis and algorithmic-conceptual axis (Modir et al., 2019). This creates four quadrants: physics-conceptual, physics-algorithmic, mathematics-conceptual, and mathematics-algorithmic. The researchers mapped three types of students physmatic difficulties in reasoning:

(a) transition errors–manifested when a student encounters difficulty in shifting between two quadrants in a given problem,

(b) displacement errors–when a student places a given problem belonging to one quadrant (say, physics-conceptual) in another quadrant (say, mathematics-algorithmic), and

(c) content errors–denote instances where the student correctly situates the problem in the designated quadrant but fails to utilize the appropriate resources or methods for its solution.

While this model was originally designed to address students' difficulties in quantum physics problem-solving in higher education, its potential for broader application led us to incorporate it into the development of some of our questionnaire's distractors. Another model (Gifford & Finkelstein, 2020) presents a categorical framework for mathematical sense-making in physics, which helps categorize various sense-making modes and understand student reasoning.

In the literature, there are also studies on specific physmatic difficulties.

For example, Planinic et al. (2019) found that students who had studied kinematics demonstrated difficulties in transferring their mathematical knowledge of graphs to physics, particularly in interpreting slope and area in kinematic contexts.

Despite the advancements in the field of physmatic models, these models create a general framework for physmatic difficulties. Additionally, these models primarily target university-level physics students while there is a shortage of empirical research on high school students. Although some specific physmatic difficulties have been addressed in the literature, they have not been extensively explored, highlighting the need for further examination.

In addition, the examination in juxtaposition of thinking approaches related to different real-world contexts seems to be a novelty of the current study. We refer to such thinking approaches as the harnessing of certain cognitive schema when performing on a specific task. To a certain sense, this follows what Flavell (1979) described in his theory of 'cognitive monitoring' as the phenomena of monitoring goals or tasks and actions or strategies.

The Research

In this study we challenged students' mathematical thinking in three categories in three contexts: purely mathematics, physics and economics. The categories were based on our mentioned above previous research and addressed:

(I) performing arithmetic operations,

(II) the meaning of zero result of a product of three factors, and

(III) sensitivity to the region of applicability of a formula.

As discussed previously, these categories challenge students in three levels of difficulties related to algorithmic thinking, providing meaning to extra-mathematic equations and comprehending the structural interwoven of mathematics and real-world disciplines. We selected these three categories based on their connection to different skills and their distinct levels of focus on mathematics versus physics (Levi et al., 2024; Pospiech & Geyer, 2022). Category I served as a baseline to identify students' basic mathematical difficulties. It also primarily involves mathematization and manipulation skills and largely ignores physical considerations. Category II reflects the challenge of interpreting symbols differently in mathematics and physics despite the similarity of operations, aligning with interpretation and validation skills. Category III addresses the constraints that physics imposes on mathematical procedures, requiring students to validate the conditions under which a mathematical operation remains applicable. Additionally, we chose difficulties that, when visualized along the physmatic scale, are positioned at significantly different points, allowing us to gain a broader range of insights.

Research Questions

1. How does the context (pure mathematics, physics, or economy) and students' difficulties in the three categories mutually impact each other?
2. What are the thinking approaches that can be related to these difficulties in view of the three contexts mentioned above?

METHOD

Tools

The questionnaire

We adopted here a quantitative diagnostic methodology, and considering the aforementioned three categories I-III, we developed a questionnaire with the objective of discerning between difficulties originating from each of the three contexts: PC, economy context (EC), and mathematics context (MC). In each of the three contexts we developed three questions–one for each of three categories I-III. Thus our questionnaire included in total nine questions (three contexts × three categories). The questions were shuffled into three different orders, diminishing potential learning effects.

In the PC we deliberately selected the everyday phenomenon of pressure in static fluids, which is outside the regular science curriculum in the country of research. This enabled mitigating a bias due to students' possible differences in prior knowledge in physics (they study science in different order and pace at their schools and classes). Economy is not studied at our schools at all. The mathematical structure was the same in each of the three contexts: $y - y_0 = a \cdot b \cdot x$. To ensure that the students understand the scenario addressed in each question, we added a brief explanation of the meaning of the symbols included in it. Although the material presented to the students was unfamiliar to them, the physmatic difficulties they encountered are linked to physmatic skills that they had already encountered in class. These skills are an integral part of what the teacher aims to develop during physics lessons and had been introduced to them in previous lessons prior to completing the questionnaire. For example, in learning Snell's law, students have dealt with cases where formulas are not valid (one cannot apply it if the incident light ray travels from higher refractive index medium into lower refractive index medium, in an angle higher than the critical angle). Another example is applying formulas for constant velocity or constant acceleration in problems in which this is not the case. Therefore, while the questions do not allow for direct conclusions about the students' understanding of the specific physics content presented, they do provide insight into their ability to apply these previously encountered skills.

In **Table 2**, we present the three sets of questions in the three contexts: Q_{I} is a set of questions concerning arithmetic operations, Q_{II} is a set of questions concerning mathematical generalization of a zero value of a product, and Q_{III} is a set of questions concerning the region of applicability of a formula. Although the literature refers to equations, in the questionnaire development, we provided the equation as a formula. Therefore, we define QIII in relation to a formula rather than an equation.

Table 2. The data collection tool: A summary of questions in physics, mathematical and economic contexts, in three categories of difficulties

Question category	Physics context	Mathematical context	Economic context
Q_I: Arithmetic operations	The formula that describes the pressure as a function of the depth underwater (h) is: $P = \rho \cdot g \cdot h + P_0$. It is also given that the water density is $\rho = 1{,}000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$, $g = 10\ \frac{\mathrm{m}}{\mathrm{s}^2}$ and the atmospheric pressure is $P_0 = 101{,}000\ \frac{\mathrm{N}}{\mathrm{m}^2}$. What should be the depth where the pressure will be $121{,}000\ \frac{\mathrm{N}}{\mathrm{m}^2}$? 1. 2 m 2. 18,990 m 3. 0.00012 m 4. 22.2 m 5. Other: ___	The formula that describes Y as a function of x is: $Y = a \cdot b \cdot x + Y_0$. It is also given that the parameter a is: $a = 10$, the parameter b is $b = 1{,}000$ and the parameter Y_0 is $Y_0 = 103{,}000$. What should be the value of x, when Y is 123,000? 1. 2 2. 18,990 3. 0.0001 4. 22.6 5. Other: ___	The formula that describes the daily total costs for producing short trousers in a clothing factory as a function of the quantity of produced units (x) is: $H = a \cdot b \cdot x + H_0$. It is also given that the cost of production is a=15 $ per unit, and for short trousers b = 10 per day, and the fixed costs of the factory are $H_0 = 2{,}000$ $ per day. What should be the number of produced units when the daily total costs are 122,000 $? 1. 800 units 2. 119,975 units 3. No unit was produced 4. 827 units 5. Other: ___
Q_{II}: Mathematical generalization of a zero value of a product	The formula that describes the pressure as a function of the depth underwater (h) is: $P - P_0 = \rho \cdot g \cdot h$, where P represents the pressure at a certain depth, h; P_0 represents the air pressure measured above the water surface; ρ represents the water density; and g is the free fall acceleration. At a certain point, it was found that the value $(P - P_0)$ is zero. Which of the following factors can be zero? 1. ρ 2. g 3. h 4. P 5. Any of the quantities in answers 1, 2, or 3 can be zero	The formula that describes Y as a function of x is: $Y - Y_0 = a \cdot b \cdot x$, where Y represents the function value at a certain x; Y_0 represents a parameter; a represents another parameter; and b represents another different parameter. It is given that the value $(Y - Y_0)$ is zero. Which of the following factors can be zero? 1. a 2. b 3. x 4. Y 5. Any of the quantities in answers 1, 2, or 3 can be zero	The formula for daily total costs of producing short trousers as a function of the quantity of produced units is: $H - H_0 = a \cdot b \cdot x$, where H represents the daily expenses to a certain amount of x; H_0 represents the daily fixed costs; a represents a constant associated with the location of the factory; and b represents a constant associated with the size of the store. At a certain day it was found that the value $(H - H_0)$ is zero. Which of the following factors can be zero? 1. a 2. b 3. x 4. H 5. Any of the quantities in answers 1, 2, or 3 can be zero
Q_{III}: Region of applicability of a formula	The formula that describes the pressure in a pool as a function of the depth underwater (h) is: $P = \rho \cdot g \cdot h + P_0$. The pressure is measured. It is also given that the water density is $\rho = 1{,}000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$, $g = 10\ \frac{m}{s^2}$ and the atmospheric pressure is $P_0 = 101{,}000\ \frac{\mathrm{N}}{\mathrm{m}^2}$. The measurement outcome was $100{,}000\ \frac{\mathrm{N}}{\mathrm{m}^2}$. What can we say about the measurement location in this case? Note: Depth is measured from the surface of the pool, and the positive axis is set downwards. 1. The depth was -0.1 m 2. The depth was 0.1 m 3. One cannot infer the water depth by using the formula with the given numerical values. Other: ___	The formula that describes Y as a function of x is: $Y = a \cdot b \cdot x + Y_0$. The value of Y is calculated. It is also given that the parameter a is: $a = 10$, the parameter b is $b = 1{,}000$ and the parameter Y_0 is $Y_0 = 103{,}000$. The calculated value of Y was 100,000. What can we say about the value of x in this case? 1. The value of x is -0.3 2. The value of x is 0.3 3. One cannot infer the value of x by using the formula with the given numerical values. Other: ___	The formula that describes the daily total costs of producing suits as a function of the quantity of the produced units (x) is: $H = a \cdot b \cdot x + H_0$. A measure for the daily total costs is calculated on a particular day. It is also given that the cost of production is a = 15 $ per square meter, and b = 10 square meters per suit, and the fixed costs of the factory are $H_0 = 130{,}000$ $ per day. The daily total cost was 100,000 $. What can we say about the number of produced units in this case? 1. The factory produced -200 units 2. The factory produced 200 units 3. One cannot infer the number of produced units by using the formula with the given numerical values. Other: ___

Table 3. Classification of answer choices by the three categories in the questionnaire

Number of options	Q_I (arithmetic): Types of error	Q_{II} (zero-product): Types of real-world error	Q_{III} (formula's region of applicability): Types of real-world error
1	Correct answer	PC: $g = 0$ EC: $a = 0$	Calculated answer
2	Subtracting a and b instead of dividing them	PC: $r = 0$ EC: $b = 0$	Absolute value of the calculated result
3	Dividing Y by Y_0	PC: $h = 0$ (correct) EC: $x = 0$ (correct)	None reasonable result (correct)
4	Transferring terms without changing their sign	PC: $P = 0$ EC: $H = 0$	-
5	-	PC: Either g or r or $h = 0$ EC: Either a or b or $x = 0$	-

We gave special attention to adjusting the questions with similar length and wording.

Design Process

The initial version of the questionnaire was developed to address difficulties in the three mentioned categories that we observed while analyzing video-recorded physics lessons (Levi et al., 2020, 2021). These were designed as PC questions. The distractors were devised based on possible reasoning paths based on the aforementioned models and the authors' experience in teaching physics and mathematics (**Table 3**).

In category Q_I, the distractors were designed to reflect various common mistakes students might make when isolating a variable in an equation. We included 'other' as an option to account for any additional mathematical errors. In all three question types–PC, MC, and EC–the distractors are parallel, although the examples provided demonstrate the distractors in the MC question only.

In the MC question, we selected option 1, which corresponds to the value 2, as the correct answer to maintain consistency across MC and PC contexts. To ensure the realism of the real-world contexts, we designed the correct answers to be reasonable within their respective scenarios. In the PC context, a depth of 2 meters is a plausible answer, while in the EC context, production quantities in the range of hundreds of units are expected; therefore, we set the correct answer to 800 units. Notably, the distractors were derived from common arithmetic errors, which sometimes led to unrealistic values.

In category Q_{II}, focusing on MC, there is a situation where a product of three quantities equals zero. Consequently, each component in the product could potentially have a zero value. In contrast, in the real-world contexts of physics and economics, not all quantities can be zero. For example, in physics, students needed to recognize that parameters like ρ and g cannot attain a zero value in the given scenario, whereas the variable h can. The fifth distractor represents purely mathematical thinking, where students might automatically assume that if the product is zero, each quantity can be zero. We placed it as distractor #5 to make sure that students read all other distractors before choosing this one automatically. The first two distractors highlight confusion about the values of ρ and g, respectively, while the fourth distractor reflects misunderstanding of the value of P_0. This demonstrates partial physical reasoning as students consider the variables rather than automatically selecting the fifth distractor. The third option, h, is correct. Hence, there are three forms of possible sources for physmatic difficulties: completely mathematical (distractor #5), partially physical (distractors #1, 2, 4), and completely physical (the correct answer #3). Similarly, the construction of distractors in the EC question follows a parallel pattern regarding understanding within the economic context.

Our design may be related to the model of Modir et al. (2019): Question Q_{III}, in its PC form, promotes physics-algorithmic thinking and perhaps mathematics-algorithmic thinking, although solving it requires physics-conceptual thinking. To solve it, one must transition from physics-algorithmic to physics-conceptual thinking. Therefore, anyone who chooses distractor #1 (the calculated answer) has made a 'displacement error' by being in the wrong (mathematics-algorithmic) quadrant of the model. Those who choose distractor #2 may also have a 'transition error' because, instead of changing to physics-conceptual, they remain in physics-algorithmic. We included 'other' as an option to account for any additional errors, although we found none.

Reliability and Validity

To ensure the questionnaire's reliability and validity, it was presented to five experts in physics education research and discussed among them. Insights from these discussions played a crucial role in refining the questionnaire. For example, to prevent misunderstandings of the Q_{III} set of questions and encourage students to consider alternative scenarios, the question was changed from "What would be the depth of the measurement (in the described situation)?" to "What can we say about the measurement location in this case?" This modification was intended to challenge students to consider the meaning of a negative result they get.

Another example, addressing students' comprehension of formula's region of applicability, is a modification of the statement in the correct answer from "the formula is not valid" to "one cannot infer the water depth by using the formula with the given numerical values". These modifications (and others) aimed to enhance the questionnaire's clarity and the similarity between the contexts of PC, MC, and EC.

We took two measures to ensure that the students understood the questions presented to them and validate our prior assumptions concerning the reasoning in making their choices.

First, we added an open response to each of the questions (in category I and category III there is an option of "other", and in category II the students were asked to explain their choices). In category III, in case where students wrote that it may be a measurement or calculation error, we explicitly considered this alternative explanation and treated such responses as correct in our analysis. Second, we conducted 10 'think aloud' interviews with randomly selected students. These interviews provided an opportunity for the students to solve the problems and offer detailed explanations for their choices. This resulted in minor language adjustments of some questions. No other reasoning to our previous assumptions was found. For the full questionnaire, see **Table 2**.

Population

We distributed the questionnaire among 199 10th-grade students (15-16 years old) who study algebra-based physics across 38 classes in 30 high schools within the research country between the years 2022 and 2024. These students are neither classified as gifted nor struggling. The students volunteered to answer the questionnaire anonymously and in their free time. The research received ethical approval from our institutional review board (IRB) (reference number: 2023Y2803). Informed consent to participate in this study was obtained from the legal guardians of all participants, in accordance with the ethical standards of our IRB.

Analysis

To address how specific real-world situations relate to students' physmatic difficulties, we compared the responses distributions corresponding to the distractors across the different contexts (PC, MC, and EC) within question sets of each category (I, II, and III). We also compared correct and incorrect answers distributions across the contexts. Chi-square tests were used to identify significant differences.

To further explore the role of the context in the aforementioned three categories of physmatic difficulties, we characterized each student by their his or her success (+) or failure (-) in each context. So, $P^-M^+E^-$ for example, represents a student encountering challenges in physics and economics yet displaying proficiency in the pure MC.

Finally, we compared respondents across different difficulty categories to investigate links between arithmetic difficulties (category I) and other physmatic difficulties (category II and category III).

FINDINGS AND FIRST INTERPRETATION

In the findings section, we will first present the whole sample analysis, dividing it into two parts: the distribution of responses across the three contexts and in each category of difficulty. Then, we will present each category of difficulty separately, discussing how context acts as a mitigating or complicating factor in category I, and the effects of context and thinking approaches in category II and category III.

The Whole Sample Analysis

Distribution of response across the three contexts

Table 4 presents the distributions of the students' chosen distractors responses across the three contexts for each of the three categories I-III and of the correct-incorrect responses in these categories (Chi-square levels of significance are indicated). A more detailed analysis for each of the categories is illustrated in **Figure 1**, **Figure 2**, and **Figure 3**.

It is apparent that in category II and category III the students' responses and correct answers distributions

Table 4. Levels of significance for Chi-square tests: Comparing students' chosen distractors across three contexts for each question category and correct-incorrect responses

	Category I		Category II		Category III	
	Responses distribution	Correct-incorrect	Responses distribution	Correct-incorrect	Responses distribution	Correct-incorrect
PC vs. EC	$p = 0.996, n.s$	$p = 0.827, n.s$ T/F = 2.317	$p = 0.332, n.s$	$p = 0.170, n.s$ T/F = 0.525	$p = 0.127, n.s$	$p = 0.300, n.s$ T/F = 0.340
MC vs. PC	$p = 0.255, n.s$	$p = 0.040^*$ T/F = 2.827	$p < 0.001^*$	$p < 0.001^*$ T/F = 1.062	$p < 0.001^*$	$p < 0.001^*$ T/F = 904
MC vs. EC	$p = 0.397, n.s$	$p = 0.066, n.s$ T/F = 2.902	$p < 0.001^*$	$p < 0.001^*$ T/F = 1.211	$p < 0.001^*$	$p < 0.001^*$ T/F = 0.990

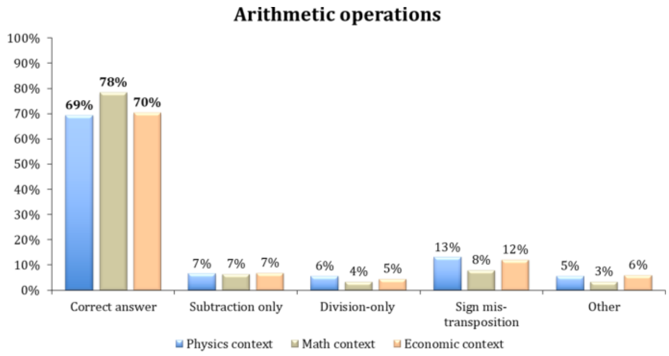

Figure 1. The distribution of responses across different contexts within the Arithmetic Operations category (N = 597 responses across three contexts for the category I question). The correct answer is highlighted in **bold** (Source: Authors' own elaboration)

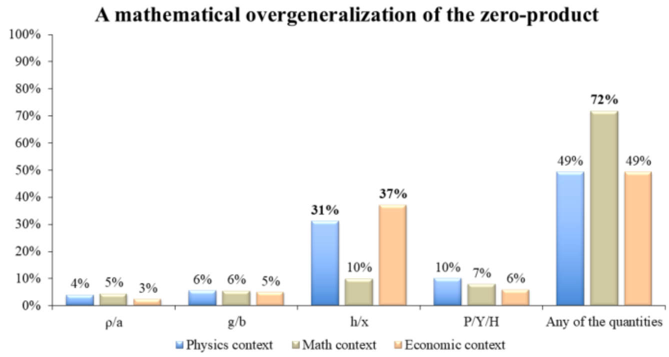

Figure 2. The distribution of responses across different contexts within the mathematical overgeneralization of the zero-product category (N = 597 responses across three contexts for the category II question). The correct answer is highlighted in **bold**) (Source: Authors' own elaboration)

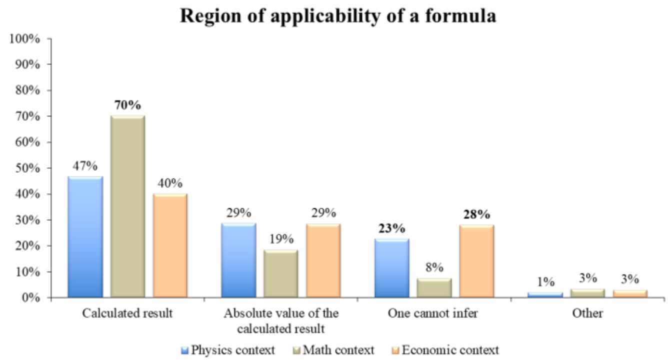

Figure 3. The distribution of responses across different contexts within the region of applicability of a formula category (N = 597 responses across three contexts for the category III question). The correct answer is highlighted in **bold**) (Source: Authors' own elaboration)

are rather similar in the two real-world contexts (PC and EC).

Students tend to perform significantly better in the MC compared to both real-world contexts. No significant differences between the three contexts were detected in category I. Note that in category II and category III the correct answers to the questions in both PC and EC are corresponding and differ from the correct answer for MC (see **Table 2**).

We infer that providing meaning to real-world formula (category II) and relating to the fundamental interrelations between mathematics and physics disciplinary structure (category III) related to the applicability boundaries of a formula (we found no indication of such consideration in economics) present a higher hurdle to the students than following a mathematical algorithm (category I).

Importantly, while the context itself (PC or MC) did not provide an extra challenge in category I in which algorithmic thinking suffice, such thinking approach was inadequate to address the other two categories. This is further supported by the analysis of the correct responses distributions

We organized the responses by success or failure in each context in each of the categories defined above. Students who answered correctly across all contexts ($P^+M^+E^+$), are distributed differently among the three categories: 55% answered correctly all contexts in category I, 20% in category II and only 8% in category III. This also means that category I (arithmetic operations) was the least challenging, when category II (mathematical overgeneralization) is somewhat more challenging, and category III (formula region of applicability) being the most challenging. We may infer that in terms of students' difficulties, the three categories present three levels of phys-math (and eco-math) thinking.

Further details can be found in **Appendix A**.

Context effect in arithmetic operations (category I questions)

We defined the context as a mitigating factor if students failed to answer the MC questions correctly but succeeded in the PC or EC questions. Conversely, the additional context was considered a complication if students answered correctly in the MC questions but incorrectly in the PC or EC questions. Thus, for students who answered correctly or incorrectly in all contexts, context does not seem to complicate or mitigate the problem. We refer to students who provided correct answers across all contexts ($P^+M^+E^+$) in category I as 'a-contextual math students,' highlighting their ability to solve mathematical problems regardless of the type or presence of context. They focus solely on mathematical operations, unaffected by whether the problem is framed within a physical, economic, or pure math context. Note that this does not ensure their success in handling questions that required higher phys-math thinking (category II and category III). This is an important evidence for those who teach physics, not to rely on better math proficiency per se in assisting their students with their physmatic difficulties.

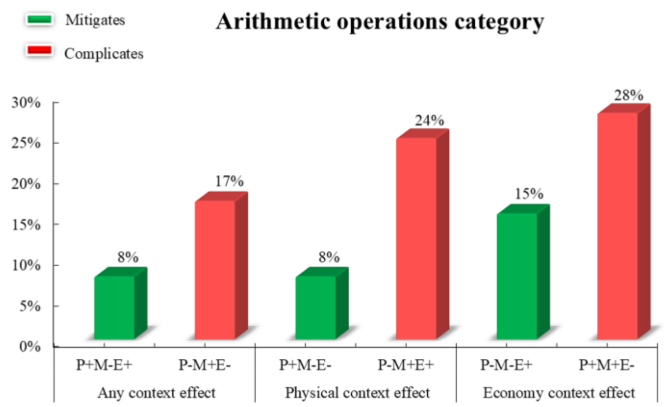

Figure 4. Responses to Arithmetic Operations difficulties category (N = 65) (Source: Authors' own elaboration)

Based on our previous finding concerning the impact of the context on students' answers, we examined the responses of the students who neither answered correctly nor incorrectly in all contexts (N = 65; 33% of the whole sample). We categorized the responses of these students according to this definition (**Figure 4**). For example, $P^-M^+E^+$ (24% of this group) suggests that physics seemingly complicates complicated the problems for some students, while P+M-E- (8% of this group) suggests that physics mitigated it for others. Note that in these examples EC neither mitigated nor complicated the MC problems.

We found that context (PC, EC, and both) tends to complicate rather than mitigate in-context arithmetic operations ($\chi^2(5) = 13.554$, $p = 0.019$)

Context effect in interpreting zero-product result (category II questions)

Whereas the set of questions in the previous category I checked whether adding real-world contexts mitigates or complicates the arithmetic operations, this set of questions in category II challenged students' awareness of the constraints posed by real-world context, not by mathematics, on interpreting the result of a mathematical expression. The set of questions in this category required the students to attribute special meaning to the real-world symbols.

In our case, a zero result of a product means that while in mathematics each of the quantities in a product can have a zero value but in the real-world contexts (PC or EC), only the variables (**h** or **x**) can have a zero value but not the parameters ρ or the near-earth calculated g (see **Table 2**).

We found that the students' thinking approach context awareness could be categorized into four groups:

1. **Students attentive to any context (AC+):** This group included P+M+E+ and P+M-E+ students (N = 44) that correctly solved the question when real-world context (physics and economics) was present, irrespective of correctness of their answer to the MC question. These students exemplified attentiveness to the context encompassed in the question and to the constraints it bears.
2. **Students inattentive to any context (AC-):** This group included P-M+E- and P-M-E- students (N = 106) that failed in the context-based questions (PC and EC), irrespective of the correctness of or incorrect their answer to the MC question.
3. **Students attentive to the physics context (PC+):** This group included P+M+E- and P+M-E- students (N = 18) that answered correctly in the PC question but not in the EC question.
4. **Students attentive to the economic context (EC+):** This group included P-M+E+ and P-M-E+ students (N = 31) that answered correctly in the EC question but not in the PC question.

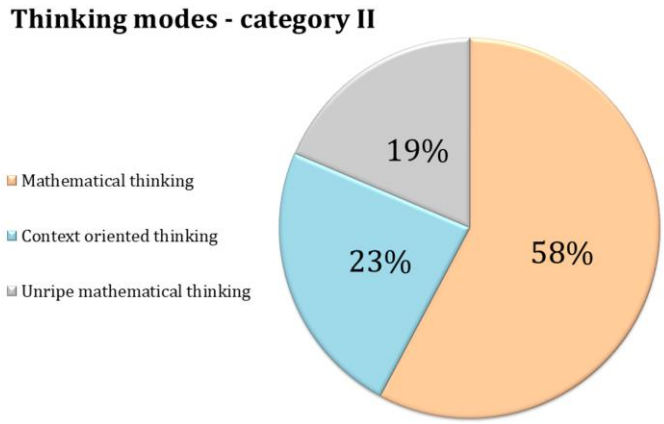

Figure 5. Thinking approaches in the category of mathematical overgeneralization of the zero-product category (N = 199) (Source: Authors' own elaboration)

The finding that the group of students who are inattentive to real-world context is much larger than the group of those who are attentive to it supports our previous observation that the context (in category I questions) was more challenging than mitigating.

The students in our sample were not completely guessing (with $\chi^2(3) = 91.593$, $p < 0.001$), which may imply that they employed certain thinking tools when they approached the questions in category II. We have found that some students overlooked real-world constraints and interpret the zero result mathematically as if any of the quantities in the product can equal zero. We regard the thinking approach that these students exhibit as 'mathematic thinking approach'.

Most students (58% of the respondents, see **Figure 5**) employed a mathematical thinking approach in at least one of the real-world problems. A substantial portion of this group (80%) encountered difficulty in all contexts, implying that these students' strong inclination towards pure mathematical thinking hindered their in-context mathematical thinking. This may warn against teaching the mathematics required for physics and physics itself independently.

Other students, who showed good mathematical thinking, did not operate it automatically, possibly influenced by the context. We refer to this approach as context-oriented thinking (not necessarily correct). Context-oriented thinking was employed by almost fourth of the students (23%) who correctly answered MC questions but in PC and EC questions chose a single quantity (not necessarily the correct one). These students did not apply mathematical thinking automatically but rather adapt their approach according to the context and sought for specific values that fit it. We included in this type of thinking $P^+M^+E^+$ students as well as $P^-M^+E^-$, in which the students chose for instance, 'ρ' in PC and 'a' in EC to equal zero. 83% of this group members exemplifying attentiveness to any context (PC, EC, and MC). A small subgroup (11%) within this category experienced difficulty in the contexts of PC and EC, as they opted for a single quantity that was not the correct one (**Figure 6**). Apparently, all members of this group were engaged with the context of the text rather than relying solely on their mathematics.

To validate our interpretation, we thoroughly examined these students' reasoning through their answers to the open questions. We found that they consistently employed mathematical reasoning in MC questions and context-based reasoning in PC and EC questions. For example, a student who chose $P = 0$ in the PC question (see **Table 2**) provided a rationale: "g is a constant, and the density cannot equal zero". This response shows that the students harnessed incomplete insights from physics and assimilated them mathematically. About 80% of this group used context-oriented thinking in both real-world contexts, not just in one of them. This supports regarding their thinking as being influenced by their context knowledge.

A considerable number of students (19%) thought that in mathematics not all quantities can equal zero but just one (this, regardless of their responses in the real-world contexts). Since these students were successful in other mathematical questions in the questionnaire survey, we consider their thinking approach as unripe mathematical thinking.

Students in this group did not necessarily fail questions with context. A considerable number of them (40%) answered correctly the EC questions but not the PC questions, and only a few (14%) answered correctly in both contexts.

We infer that students in this group struggled with combining mathematics and in-context mathematics especially in physics. In most cases these students did not provide reasons for their choices in the open question and in the interviews thus limiting any further interpretation.

Further details on the categorization process can be found in **Appendix B**.

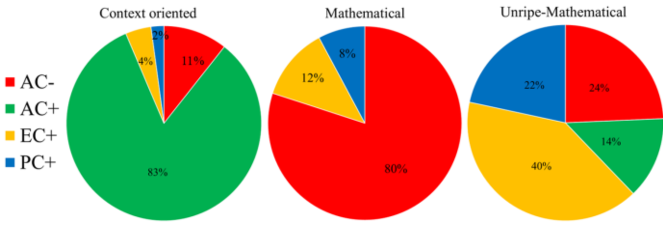

Figure 6. Context effect of different thinking approaches, N=199 (context-oriented thinking (N=47), mathematical thinking (N=115) and unripe or missing mathematical thinking (N=37)) in the category of the difficulty of mathematical overgeneralization of the zero-product (Source: Authors' own elaboration)

In order to further investigate the effect of the context on students' thinking, we examined the distribution of the context effect within each type of thinking and looked for possible associations between them (**Figure 6**). The Chi-square statistics revealed significant variations in the responses across different contexts within each type of thinking ($\chi^2(6\,,199) = 131.37, p < 0.001$). These results indicate a connection between the context and thinking approaches. However, it is not possible to determine whether the context influences the thinking approach or whether the thinking approach when solving a question in a given context leads to success or failure.

Context effect in category III: Region of applicability of a formula

We analyzed this category in a similar manner to the former category, addressing the context effect and thinking approaches (see **Appendix B**).

The results here resemble those observed in category II in terms of the difference between the students' context attentiveness (AC- = 58%; AC+ = 12%; EC+=18%; PC+ = 12%). Here too the students were not guessing ($\chi^2(3) = 122.99$, $p < 0.001$).

The results regarding the types of thinking are similar to what we observed in category II.

The students that harnessed mathematical thinking (62% of the respondents) (**Figure 7**) accepted the calculated result as is, without considering that the variables are limited to positive numbers in the real-world contexts. The majority of students in this group (72%) found PC and EC to be challenging. This can imply that their mathematical mindset impeded them from considering the question's context.

By contrast, students who adopted context-oriented thinking (21% of the respondents). We included in this category those that correctly answered the MC question and also chose the correct answer or the absolute value of the calculated result in real-world questions (see **Table 2**). These students understand that the result (depth or number of units) cannot be negative. A considerably large group of this group (38%) demonstrated attentiveness to any context, and chose the

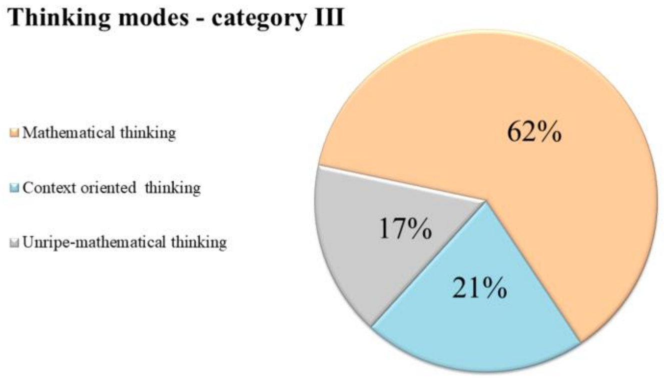

Figure 7. Students' thinking approaches in the category region of applicability of a formula category (N = 199) (Source: Authors' own elaboration)

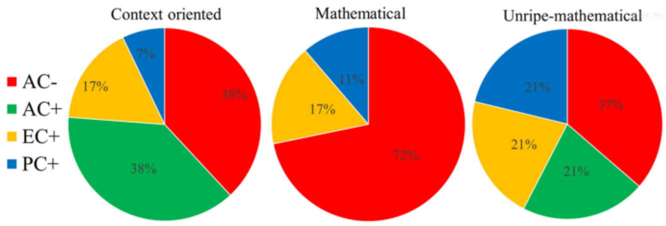

Figure 8. Context effect of different thinking approaches (context-oriented thinking (N=42), mathematical thinking (N=124), and unripe or missing mathematical thinking (N=33)) in the category region of applicability of a formula category (Source: Authors' own elaboration)

correct answer in PC and EC. A similar-sized group (38%) within this type of thinking chose the absolute value of the calculated result in PC and EC.

Less than fifth of students (17%) exhibited unripe mathematical thinking. Students in this group tend to select incorrect answers more often in PC and EC contexts (37%) than they select correct answers in the two real-world contexts (21%).

Table 5 summarizes the thinking approaches analysis in category II and category III (**Figure 8**). Although the differences between category II and category III are not statistically significant, the findings may imply that students tend to rely more on their mathematics thinking in category III than in category II. This is left for further research.

Connections Between Category I and Categories II and III

As a further investigation we examined possible association between students' proficiency in mathematics, according to their arithmetic skills (category I), and the types of thinking they harness in the two higher challenges of category II and category III (see **Table 6**).

First, we found significant differences in students' responses to the questions of the higher categories, depending on their responses in category I questions. A-contextual math students (P+M+E+, students with high mathematical proficiency in category I; 55%) demonstrated more context-oriented thinking and exhibited less unripe mathematical thinking in category II and category III compared to all other students, In this respect, it is obvious that competency in mathematics assists students in addressing higher demanding context focused questions.

Still, twice more of the a-contextual students the mathematical thinking in category II, than those in this group that exhibited context-oriented thinking. This tendency is even more pronounced in category III. We thus infer that students with high competency in math, although perform better than other students, still refrain from harnessing a different approach–context oriented thinking–required for answering questions of the higher categories.

While the percentages representing thinking approaches may not indicate the same students, and some students may adopt one thinking approach in category II and another in category III, the similar percentages still suggest a comparable prevalence of each thinking approach that a teacher can expect to see in their class across both category II and category III.

Table 5. Context attentiveness and thinking approach in category II and category III

	Category II	Category III
Context attentiveness	AC^{+} = 22% & AC^{-} = 53%	AC^{+} = 12% & AC^{-} = 58%
Mathematical thinking approach	58%	62%
Context oriented thinking approach	23%	21%
Unripe mathematical thinking approach	19%	17%
Variations across different contexts	$\chi^2(6, 199) = 131.37, p < 0.001$	$\chi^2(6, 199) = 51.514, p < 0.001$

Table 6. Students' proficiency in mathematics according to category I and types of thinking in category II and category III

Responses in category I	Types of thinking represented in category II*	Types of thinking represented in category III**
A-contextual (55%)	Mathematical = 65%	Mathematical = 73%
	Context-oriented = 30%	Context-oriented = 24%
	Unripe = 5%	Unripe = 3%
Others (45%)	Mathematical = 48%	Mathematical = 49%
	Context-oriented = 17%	Context-oriented = 17%
	Unripe = 35%	Unripe = 34%

Note. *$\chi^2(2) = 28.5$, $p < 0.001$ & **$\chi^2(2) = 34.1$, $p < 0.001$

DISCUSSION AND CONTRIBUTION TO LITERATURE

Our findings provide new insights into the interplay between mathematical and real-world thinking in physics education, revealing distinct student thinking approaches. In this section, we discuss these findings in relation to prior research, highlighting their implications for understanding physmatic difficulties.

The Relationship Between Students' Physmatic Difficulties and Context

Regarding arithmetic operations (category I), our findings support previous reports (Carli et al., 2020; Macuca et al., 2022) according to which real-world context thinking skills are closely intertwined with mathematical thinking skills.

Most of the students in our study (55%; see **Figure 4**) successfully tackled the arithmetic question in all types of contexts: PC, EC, and MC. Thus, unsurprisingly, arithmetic abilities support many students in solving problems in Real-world context. However, for the remaining 45% of the students a real-world context introduces difficulty beyond the inherent complexity of the mathematical question.

Regarding mathematical overgeneralization of zero-product result and region of applicability of a formula (category II and category III), the findings indicate that while the majority of our addressees performed very well in a purely mathematical context, many of them exhibited difficulties in real-world context, most of them apply mathematical thinking in an overarching way. We infer that for those students triggering mathematical thinking approach inhibits real-world considerations familiar to them such as that water density cannot equal zero.

Mathematical Transfer and the Influence of Context

The transfer of knowledge from mathematics to physics has been shown to be effective in solving algebraic problems. while the reverse transfer is not helpful (Bassok & Holyoak, 1989; Rebello et al., 2017). Our results support these findings regarding category I and show that most of those who had previous mathematical knowledge (i.e., most of our population) could successfully solve the physical problem (with unfamiliar content). However, we found that transfer from mathematics is not always beneficial, as it might cause mathematical overgeneralization, as illustrated in questions concerning overgeneralization of the zero-product or region of applicability of a formula (category II and category III). These findings are consistent with previous research (Ivanjek et al., 2016; Planinic et al., 2019). For physics teachers this implies that resting on good math teaching is insufficient–they have to support their students' physmatic thinking in their teaching.

We found respondents who had difficulty in questions that included PC and/or EC but had no difficulty in the pure MC in the three categories:

(I) context arithmetic difficulty (23%),

(II) overgeneralization of the mathematical zero-product property (55%), and

(III) unawareness of the region of applicability of a formula (63%).

For these students mathematics skills provided no support in addressing real-world cognitive challenges (Levi et al., 2024) (**Appendix A**). These findings can contribute to the understanding that as the need for physmatic skills increases, the difficulty of transfer also rises.

Thinking Approaches and the Relationship to Difficulties and Context

Our study suggests that the cognitive roots of physmatic difficulties may stem from the thinking approaches of students that are influenced by the context. By thinking approach, we mean the harnessing of certain cognitive schema when performing on a specific task.

We identified three thinking approaches among the students: context-oriented thinking, mathematical thinking, and unripe or missing mathematical thinking. It is noteworthy that the distribution of students among the thinking approaches is similar in both difficulties of overgeneralization of the zero-product and region of applicability of a formula. The dominant type is mathematical thinking. Among those who demonstrated such a thinking type, PC and EC questions posed a challenge for the students. This suggests that mathematical thinking prevails over physical or economic thinking. This may be interpreted as a cognitive challenge in interpreting mathematical expressions in PC or EC due to a lack of structural skills such as validation, consistent with previous research (Karam et al., 2010; Pietrocola, 2008; Pospiech & Geyer, 2022; Uhden et al., 2012).

The next type of thinking is context-oriented thinking. Apparently, the ability to think mathematically on MC question and differently on PC or EC question contributes to success in real-world contexts. In overgeneralization of the zero-product, 83% of the students with context-oriented thinking chose the correct answer. In the region of applicability of a formula, all such students either chose the correct answer or the absolute value of the calculated answer, realizing that a negative value could not be a correct answer. Context-based thinking encourages students to use structural skills such as interpretation and validation (Karam et al., 2010; Uhden et al., 2012). This result may suggest that if we introduce students to a mindset that emphasizes attention to context, their ability to solve problems related to real-world situations can possibly improve. That is, they might better overcome transition errors (Modir et al., 2019) or more effectively make translations between different categorical structures (Gifford & Finkelstein, 2020). This inference of course remains for further examination.

One possible explanation of the higher prevalence of mathematical thinking compared to context-oriented thinking could be the curriculum in the country of research and many other countries as well, which places a strong emphasis on mathematics education, both in terms of the number of weekly lessons and the years of study, compared to science education.

Lastly, for many students characterized by the unripe mathematical thinking approach, the non-mathematical context questions were more challenging than manageable. Nevertheless, there was a non-negligible number of correct responses among these students. It is possible that some of the students who struggle with mathematics were unable to perform the necessary calculations, while others were led by their difficulties in math to results that did not match any of the distractors provided in the question. As a result, both chose the correct option 'it is impossible to know.' As we do not have access to these students' calculations, future research will be needed to distinguish between them.

Physics Context vs. Economic Context

Beyond thinking approaches, various influences of context on our respondents' answers were observed. Across all three cognitive difficulties and through all thinking approaches, students were more attentive to EC than to PC (see **Figure 1**, **Figure 2**, **Figure 2**, **Figure 7**, **Figure 8**). This, even if not found to be statistically significant, deserves some attention. While literature lacks direct comparisons between different contexts for specific mathematical questions, this possibly suggests that, beyond cognitive difficulties and thinking approaches, the specific context also affects students' thinking. It is possible that concepts from the field of economics are more intuitive for students, which might make the economic context somewhat more familiar and, therefore, less challenging for them. The greater difficulty in the PC compared to the economic context align with reports suggesting that unfamiliar contexts can hinder deep understanding (e.g., Song, 2011).

Connections Between Proficiency in Mathematics and Physmatic Difficulties

Our research shows that students with a higher proficiency in mathematics in category I demonstrated more context-oriented thinking in category II and category III, while students who struggled with technical arithmetic operations in category I also exhibited unripe mathematical thinking when required to apply physical or economic understanding and mathematical rules in context (see **Table 6**). This aligns with previous results (Sigus & Mädamürk, 2024). However, higher mathematical level does not ensure that a student will refrain from overgeneralizing mathematical concepts when it is inappropriate to do so. This result may stem from a strong tendency to cling to mathematical procedural automation, rather than considering the context and its limitations.

The Relationship Between Physmatic Difficulties, Thinking Approaches, and Teachers PCK

The PCK model for physmatic has four components (Lehavi et al., 2015) and our findings can contribute to two of them. Regarding the *knowledge of physmatic contents*, it brings attention to specific difficulties in the interplay of physics and mathematics in physics lessons and, regarding *teachers' knowledge about students* we highlight the cognitive source of students' difficulties as stemming from three different thinking approaches. We expand teachers' knowledge of assessment by offering questionnaires, like those we have developed and presented here that may enable them to assess physmatic difficulties. Practically, our findings can help teachers better understand their students' physmatic difficulties and better plan their lessons by adjusting to the different levels of each difficulty.

Implementing a diagnostic questionnaire in the classroom can be instrumental for educators in guiding their students based on their thinking patterns. Additionally, using this tool in other categories has the potential to establish a map of more physmatic difficulties to be considered by physics educators.

Limitations

Despite the possible significant contribution of the paper to the field, there are some limitations to the study.

One limitation is the relatively low respondents' percentile (about 26%), which is due to the voluntary nature of the survey. This might have biased our sample toward a higher percentile of diligent students. However, if this pattern holds among motivated students, one could assume that an even higher rate of difficulties might emerge.

The design of the questionnaire may have allowed students to make guesses, which could result in some answers not accurately reflecting their true knowledge and skills. This issue is common in many field studies, and future research focusing on in-depth interviews could help minimize this by providing a better understanding of students' decision-making processes in the current research perspectives. Another limitation is that dividing participants into groups and sub-groups based on difficulty categories results in small sample sizes within each group, making it harder to detect phenomena with smaller effects that are not necessarily the main focus of the current study. Therefore, future research with a larger sample size could help examine these relationships. Additionally, our ability to determine students' levels in mathematics, physics, or economics based on the questionnaire is limited, and this should be considered with caution.

SUMMARY, CONCLUSIONS, AND RAMIFICATIONS

In this study, we applied a questionnaire that allowed us to examine and diagnose the connections between three categories of physmatic pre-identified difficulties and their relation to students' thinking approaches in mathematical and real-world context. The study revealed a significant influence of context (pure mathematics, physics, and economics) on students' performance across three categories of physmatic difficulties: arithmetic operations, overgeneralization of the zero-product property, and the region of applicability of a formula. The impact of context varied across the three difficulty categories. Context had less impact on arithmetic problems (category I) compared to the more complex tasks involving the zero-product property (category II) and the region of applicability of a formula (category III). This means that the categories evoke different cognitive resources required for different levels of phys-math thinking. Therefore, one cannot expect that the same teaching methods will assist students in developing higher thinking levels as those that work well in the lower level.

Another important finding indicates that students employed three main approaches of thinking when tackling physmatic challenges. It would be a fair assumption that most students arrive at the physics class with unripe mathematical thinking, displaying incomplete or inconsistent integration of mathematical reasoning and real-world context understanding. The finding that mathematical thinking can prevent some students from developing a higher level of context-oriented thinking, mostly required in the higher levels represented by category II and category III, should warn physics teachers from relying excessively on students' mathematical thinking. Although students with higher mathematical proficiency (as measured by performance in the arithmetic operations category) were more likely to use context-oriented thinking in the more complex tasks, some of them still struggled to apply context appropriately. The multi-faceted nature of physmatic difficulties, highlighted by our study, suggests that focusing on single aspects, such as mathematical proficiency, might not be sufficient to address students' challenges.

The study confirms that the context significantly impacts students' ability to solve physmatic problems, suggesting that instruction should focus not only on mathematical procedures but also on their application within relevant contexts. The recognition that students use diverse thinking approaches when tackling physmatic problems is crucial for effective teaching. Instructional strategies should therefore address these differences to improve outcomes.

With regard to the comparison between physics and economics, we should note that while not statistically significant, students consistently showed more attentiveness to the EC than the PC across all difficulties and thinking approaches. Further research is needed to draw more well-founded conclusions in this respect. Further research is also needed to investigate the interactions between context, thinking approaches, and student performance in a wider range of physmatic tasks and to develop proper instructional strategies that address the identified challenges effectively.

Author contributions: HL, AM, & YL: contributed to the design of the study and participated in discussions on the background, methodology, findings, and discussion; **HL:** developed the questionnaires, collected most of the data and analyzed it, and wrote the manuscript; **AM & YL:** provided feedback over several iterations; & **AM:** assisted with data collection. All authors agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Acknowledgments: The authors would like to thank Gesche Pospiech and Lana Ivanjek for fruitful discussions and deep insights.

Ethical statement: The authors stated that the study was approved by the Ethics Committee of the School of Education at the Hebrew University of Jerusalem on 1 May 2023 with approval code 2023Y2803. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Bassok, M., & Holyoak, K. J. (1989). Interdomain transfer between isomorphic topics in algebra and physics. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 15*(1), 153-166. https://doi.org/10.1037//0278-7393.15.1.153

Carli, M., Lippiello, S., Pantano, O., Perona, M., & Tormen, G. (2020). Testing students' ability to use derivatives, integrals, and vectors in a purely mathematical context and in a physical context. *Physical Review Physics Education Research, 16*, Article 010111. https://doi.org/10.1103/PhysRevPhysEducRes.16.010111

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive developmental inquiry. *American Psychologist, 34*(10), 906-911. https://doi.org/10.1037/0003-066X.34.10.906

Fraihat, M. A. K., Khasawneh, A. A., & Al-Barakat, A. A. (2022). The effect of situated learning environment in enhancing mathematical reasoning and proof among tenth grade students. *Eurasia Journal of Mathematics, Science and Technology Education, 18*(6), Article em2120. https://doi.org/10.29333/ejmste/12088

Gifford, J., & Finkelstein, N. (2020). Categorical framework for mathematical sense making in physics. *Physical Review Physics Education Research, 16*, Article 020121. https://doi.org/10.1103/PhysRevPhysEducRes.16.020121

Gingras, Y. (2001). What did mathematics do to physics? *History of Science, 39*(4), 383-416. https://doi.org/10.1177/007327530103900401

Heck, A., & van Buuren, O. (2019). Students' understanding of algebraic concepts. In G. Pospiech, M. Michelini, B. S. Eylon (Eds.), *Mathematics in physics education* (pp. 53-74). Springer. https://doi.org/10.1007/978-3-030-04627-9_3

Hestenes, D. (1987). Toward a modeling theory of physics instruction. *American Journal of Physics, 55*(5), 440-454. https://doi.org/10.1119/1.15129

Hull, M. M., Kuo, E., Gupta, A., & Elby, A. (2013). Problem-solving rubrics revisited: Attending to the blending of informal conceptual and formal mathematical reasoning. *Physical Review Physics Education Research, 9*, Article 010105. https://doi.org/10.1103/PhysRevSTPER.9.010105

Ivanjek, L., Susac, A., Planinic, M., Andrasevic, A., & Milin-Sipus, Z. (2016). Student reasoning about graphs in different contexts. *Physical Review Physics Education Research, 12*, Article 010106. https://doi.org/10.1103/PhysRevPhysEducRes.12.010106

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Karam, R., Pospiech, G., & Pietrocola, M. (2010). Mathematics in physics lessons: Developing structural skills. In *Proceedings of the GIREP-EPEC & PHEC 2009 International Conference* (pp. 17-21).

Lehavi, Y., & Galili, I. (2009). The status of Galileo's law of free-fall and its implications for physics education. *American Journal of Physics, 77*(5), 417-423. https://doi.org/10.1119/1.3096643

Lehavi, Y., Bagno, E., Eylon, B. S., Mualem, R., Pospiech, G., Böhm, U., Krey, O., & Karam, R. (2015). Towards a PCK of physics and mathematics interplay. In *The GIREP MPTL 2014 Conference Proceedings* (pp. 843-853).

Lehavi, Y., Bagno, E., Eylon, B. S., Mualem, R., Pospiech, G., Böhm, U., Krey, O., & Karam, R. (2017). Classroom evidence of teachers' PCK of the interplay of physics and mathematics. In *Key Competences in Physics Teaching and Learning: Selected Contributions from the International Conference GIREP EPEC 2015, Wrocław Poland, 6–10 July 2015* (pp. 95-104). Springer International Publishing. https://doi.org/10.1007/978-3-319-44887-9_8

Levi, H., Merzel, A., Lehavi, Y., & Schwarz, B. (2020). Categorization of physmatic misconceptions. In *IPS Conference*. Rehovot: Weizmann Institute of Science.

Levi, H., Merzel, A., Schwarz, B., & Lehavi, Y. (2021). Students' reasoning while using mathematics in physics lessons. *3rd World Conference of Physics Education* [Hanoi: online talk].

Levi, H., Merzel, A., Lehavi, Y., & Schwarz, B. (2024). Diagnosing the cognitive source of students' difficulties within the physics-mathematics interplay context. In *Teaching and learning physics effectively in challenging times* (pp. 175-187). Springer, Cham. https://doi.org/10.1007/978-3-031-72541-8_12

Macuca, A., Poljančić, B., Jurdana-Šepić, R., & Lončarić, K. (2022). Relationship between mathematical competencies and achievements in physics of 14-year-old students. In *Proceedings of the GIREP International Conference*.

Modir, B., Thompson, J., & Sayre, E. (2019). Framing difficulties in quantum mechanics. *Physical Review Physics Education Research, 15*, Article 020146. https://doi.org/10.1103/PhysRevPhysEducRes.15.020146

Pietrocola, M. (2008). Mathematics as structural language of physical thought. *Kansas State University*. https://web.phys.ksu.edu/icpe/publications/teach2/pietrocola.pdf

Planinic, M., Susac, A., Ivanjek, L., & Milin Šipuš, Ž. (2019). Comparing student understanding of graphs in physics and mathematics. In G. Pospiech, M. Michelini, B. S. Eylon (Eds.), *Mathematics in physics education* (pp. 233-246). Springer. https://doi.org/10.1007/978-3-030-04627-9_10

Pospiech, G. (2019). Framework of mathematization in physics from a teaching perspective. In G. Pospiech, M. Michelini, B. S. Eylon (Eds.), *Mathematics in physics education* (pp. 1-33). Springer. https://doi.org/10.1007/978-3-030-04627-9_1

Pospiech, G., Eylon, B., Bagno, E., Lehavi, Y., & Geyer, M. A. (2015). The role of mathematics for physics teaching and understanding. *IL Nuovo Cimento C Geophysics Space Physics C, 38*, 110. https://doi.org/10.1393/ncc/i2015-15110-6

Pospiech, G., & Geyer, M.-A. (2022). Structural skills of students in solving physical-mathematical tasks. In C. Michelsen, A. Beckmann, V. Freiman, U. T. Jankvist, & A. Savard (Eds.), *Mathematics and its connections to the arts and sciences* (pp. 299-310). Springer. https://doi.org/10.1007/978-3-031-10518-0_16

Pospiech, G., Eylon, B., Bagno, E., Lehavi, Y., & Geyer, M. A. (2015). The role of mathematics for physics teaching and understanding. *Nuovo Cimento C Geophysics Space Physics C, 38*, Article 110. https://doi.org/10.1393/ncc/i2015-15110-6

Rebello, N., Cui, L., Bennett, A., Zollman, D., & Ozimek, D. (2017). Transfer of learning in problem solving in the context of mathematics and physics. In D. H. Jonassen (Ed.), *Learning to solve complex scientific problems* (pp. 223-246). Routledge. https://doi.org/10.4324/9781315091938-10

Redish, E., & Kuo, E. (2015). Language of physics, language of math: Disciplinary culture and dynamic epistemology. *Science and Education, 24*, 561-590. https://doi.org/10.1007/s11191-015-9749-7

Reyes, J., Insorio, A. O., Ingreso, M. L. V., Hilario, F. F., & Gutierrez, C. R. (2019). Conception and application of contextualization in mathematics education. *International Journal of Educational Studies in Mathematics, 6*(1), 1-18.

Sigus, H., & Mädamürk, K. (2024). Context matters: The importance of extra-mathematical knowledge in solving mathematical problems. *Frontiers in Education, 9*. https://doi.org/10.3389/feduc.2024.1334034

Song, M. (2011). *Effects of background context and signaling on comprehension recall and cognitive load: The perspective of cognitive load theory* [PhD thesis, University of Nebraska-Lincoln].

Taber, K. S. (2000). Finding the optimum level of simplification: The case of teaching about heat and temperature. *Physics Education, 35*(5), Article 320. https://doi.org/10.1088/0031-9120/35/5/301

Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G. (2012). Modelling mathematical reasoning in physics education. *Science and Education, 21*, 485-506. https://doi.org/10.1007/s11191-011-9396-6

Yerushalmi, E., & Magen, E. (2006). Same old problem, new name? Alerting students to the nature of the problem-solving process. *Physics Education, 41*(2), Article 161. https://doi.org/10.1088/0031-9120/41/2/007

Zaslow, E. (2005). Physmatics. *arXiv*. https://doi.org/10.48550/arXiv.physics/0506153

Zhao, F. F., & Schuchardt, A. (2021). Development of the sci-math sensemaking framework: Categorizing sensemaking of mathematical equations in science. *International Journal of STEM Education, 8*, Article 10. https://doi.org/10.1186/s40594-020-00264-x

Zhao, T., & Perez-Felkner, L. (2022). Perceived abilities or academic interests? Longitudinal high school science and mathematics effects on postsecondary STEM outcomes by gender and race. *International Journal of STEM Education, 9*, Article 42. https://doi.org/10.1186/s40594-022-00356-w

APPENDIX A

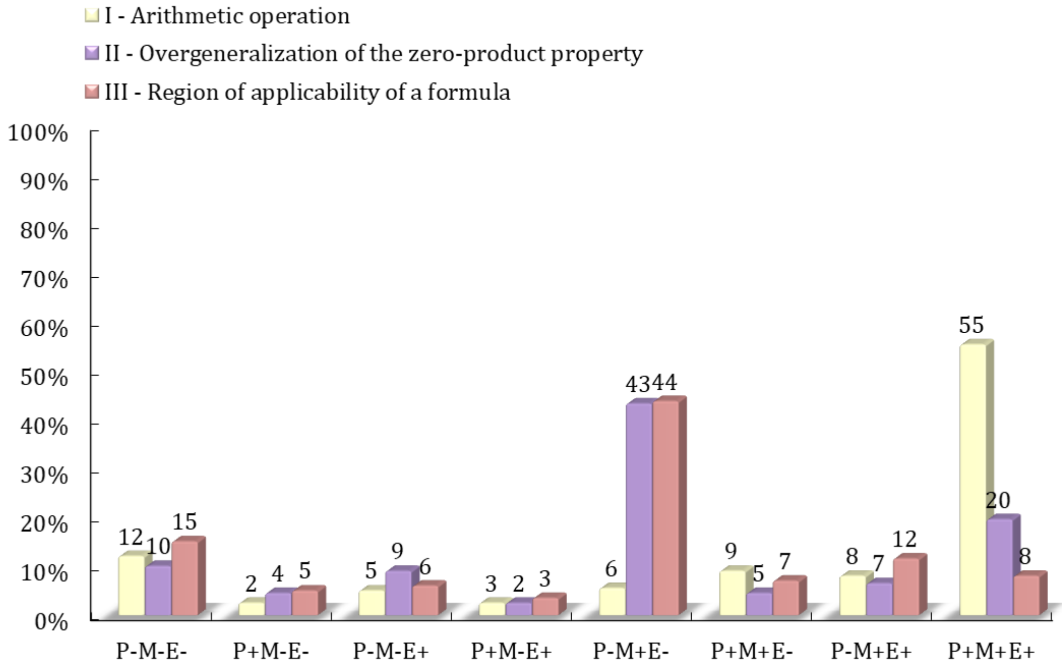

Figure A1. Percent of correct and incorrect responses rates organized by form of context responses, by categories of difficulties (N=199). '+' denotes correct response within the specific context, while '-' indicates incorrect response (Source: Authors' own elaboration)

APPENDIX B

Table B1. Categorization process in category II and category III

		MC	PC/EC	Meaning
Adaptive approach	Mathematical overgeneralization of zero-product	Option 5 (any of the quantities in answers 1, 2, or 3 can be zero)	Single value (options 1/2/3/4)	Transition from mathematical thinking in MC to thinking about a specific value in context
	Region of applicability of a formula	Option 1 (-0.3)	Options 2/3	Transition from mathematical thinking in MC to thinking about the correct answer in context
Mathematical approach	Mathematical overgeneralization of zero-product	Single value (options 1/2/3/4)	Option 5 in at EC/PC/both	Generalizing mathematical thinking in the presence of context
		Option 5 (any of the quantities in answers 1, 2, or 3 can be zero)	Option 5 in at EC/PC/both	Mathematical thinking with and without context
	Region of applicability of a formula	Option 2/other	Option 1 in EC/PC/both	Generalizing mathematical thinking in the presence of context
		Option 1 (-0.3)	Option 1 in EC/PC/both	Mathematical thinking with and without context
Unripe mathematical approach	Mathematical overgeneralization of zero-product	Single value (options 1/2/3/4)	Single wrong value (options 1/2/4)	No ability to apply the mathematical rule
		Single value (options 1/2/3/4)	Single correct value (options 3)	Able to apply the rule only in context
	Region of applicability of a formula	Single value (options 2/3/other)	Option “other” at EC/PC/both	No ability to apply the mathematical rule
		Single value (options 2/3/other)	Options 2/3 at EC/PC/both	Able to apply the rule only in context