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Enhancing geometry problem-solving through visualization for multilingual learners

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Abstract

Visualization is key in developing learners' problem-solving skills and mathematical reasoning, yet many South African Grade 11 multilingual learners (MLs) struggle with geometry. This qualitative study, informed by the commognitive framework, explores how visual aids and language influence their understanding. Eighteen MLs participated in task-based interviews and focus group discussions. Findings revealed two main challenges: limited understanding of geometry terms like "arc," "bisect," and "subtend," and difficulty visualizing concepts without a clear vocabulary. However, when visual representations supported instruction, learners showed improved comprehension, identified misconceptions, and communicated their reasoning more effectively. The study concludes that combining visual aids with explicit language support enhances MLs' geometry learning. Multilingual classrooms should incorporate annotated diagrams, drawing tasks, and vocabulary-focused prompts to strengthen visualization and conceptual understanding.

Keywords: mathematics problem-solving, visualization in geometry, multilingual learning in mathematics education, commognition in geometry thinking

INTRODUCTION

Solving mathematical problems is an important aspect and is becoming necessary in mathematics curricula worldwide (National Council of Teachers of Mathematics [NCTM], 2022; Suto & Zanini, 2018). Visualization is also a significant aspect of spatial reasoning and enhances problem-solving abilities (Lowrie & Logan, 2023; Schenck & Nathan, 2024). Visualization methods assist learners in comprehending a problem, exploring possible solutions, and developing new insights (Samosa et al., 2021). South African learners–particularly those in rural, multilingual contexts–consistently struggle with geometry problem-solving (Naidoo & Kapofu, 2020; Tachie, 2020). The first author, now in his sixth year of teaching FET-phase mathematics at a rural South African school where English proficiency is limited, observed that learners frequently misinterpret questions or lack access to essential geometric terminology. Mudhefi et al. (2024) found that South African grade 12 learners' weak conceptualization of terms such as "angle bisector" and "subtend" directly impedes their reasoning, while Naicker (2021) links low engagement and unsuitable pedagogies to persistent misunderstandings.

Existing studies (Mudhefi et al., 2024; Naicker, 2021) document what learners cannot do but rarely explain why language barriers and abstract representations interact to block understanding. According to Sfard's (2008) commognitive theory, mathematical thinking is a form of discourse: learners develop concepts by internalizing and using the specialized language of mathematics. In multilingual classrooms, learners must juggle multiple linguistic codes, which can fracture this internal communication loop and undermine their ability to create accurate mental images of geometric structures (Essien, 2013).

In rural South Africa, these challenges are compounded by scarce resources and limited access to dynamic visual aids (Duvenhage, 2020; Sao, 2008). When visuals appear only as static, unannotated diagrams, vocabulary gaps prevent learners from mapping words onto shapes. This study bridges two gaps in the literature: first, it applies a commognitive lens to explain how terminology deficits disrupt visualization; second,

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Contribution to the literature

- This paper advances the understanding of how MLs navigate the dual challenges of language and visualization in high school geometry, aiming to improve geometry problem-solving skills in multilingual contexts.
- Applying the commognitive framework highlights the interdependence of discourse and spatial reasoning in mathematical thinking.
- The study offers practical, evidence-based strategies, such as pairing visual aids with explicit vocabulary instruction, that can inform more inclusive and effective geometry teaching in linguistically diverse classrooms.

it examines how targeted visual-language interventions can restore that link.

LITERATURE REVIEW

Visualization in Mathematics Education

Zimmermann and Cunningham's (1991) early definition of visualization as "a process of constructing or using geometrical or graphical representations of mathematical concepts" laid an essential foundation for understanding how learners engage with spatial ideas. Building on this, Mudaly and Rampersad (2010) expanded the concept to include both mental imagery and external representations. In addition, Yin (2010) states that while the ability to solve problems "is at the heart of mathematics," visualization "is the heart of mathematical problem-solving" (p. 2). These foundational perspectives remain valuable, but contemporary research urges us to move beyond defining visualization toward critically examining *how* different tools shape reasoning (Ge et al., 2024; Wu et al., 2020). This study's visualization encompasses the full spectrum–from internal mental models to annotated diagrams and dynamic digital displays–each serving distinct cognitive and communicative functions.

Critical Analysis of Visualization Tools

Diagrams and static figures

Drawing on Duval's (2017) assertion that figures condense geometric situations more effectively than verbal descriptions, this review acknowledges that static diagrams remain indispensable in resource-poor contexts. However, Duval's (2017) framework benefits from a contemporary lens: when diagrams are annotated to spotlight relationships (for example, highlighting arcs alongside subtended angles), learners can negotiate vocabulary gaps and correct misconceptions more readily (Mesaroš, 2012; Mudaly, 2012).

Graphs, pictures, and symbolic notation

Graphs and pictures extend beyond pure illustration by integrating symbolic annotations, number lines, and data tables (Bautista & Ortega-Ruiz, 2015). Unlike unadorned images, these hybrid representations engage learners in translating between verbal statements and spatial configurations–an act that reinforces both vocabulary and structural understanding. Gestural scaffolding further enriches this process: Chu and Kita (2011) demonstrate that even silent hand movements can externalize learners' internal spatial reasoning and support the peer-to-peer explanation.

Manipulatives

Physical manipulatives translate abstract concepts into concrete experiences, but their effectiveness hinges on context (Baroody, 1989; Bartolini & Martignone, 2020). When learners manipulate, say, a chord-and-arc model while simultaneously naming each element, the embodied action anchors specialized terms in perceptual experience–bridging the commognitive gap between word and concept.

Digital technologies and videos

Chawla and Mittal (2013) rightly emphasize technology's dual role in shaping curriculum and enhancing understanding. Still, the review by Shoba (2020) clarifies *how* interactive simulations help multilingual learners (MLs) link terminology to motion: dragging an angle bisector and watching measurements update offers immediate feedback that static images cannot. This dynamic coupling of visual and linguistic feedback is especially powerful for learners grappling with English-based terminology in rural South African classrooms.

The role of visualization in learning in the context of geometry

According to Bruner's (1996) three models of representation, the learning process is structured into three levels: enactive, iconic, and symbolic. The enactive level is particularly significant for visualization, as it bridges practical experiences and formal understanding, effectively serving as a mediator in communication (Deliyianni et al., 2009; Sintonen, 2024). Diagrams or images that learners use or construct to enhance their understanding help form mental representations, facilitating problem-solving (Deliyianni et al., 2009;

Rösken & Rolka, 2006). Visualization not only aids in establishing relationships between mathematical concepts but provides an effective means for solving problems systematically, semantically, and pragmatically (Sintonen, 2024).

Supporting this perspective, a study by Mudaly (2016) on the role of visualization in Euclidean geometry proofs found that learners who drew diagrams during the proving process had a clearer understanding of the problems. This finding aligns with the focus of this study, highlighting the effectiveness of visualization in mathematical problem-solving, especially for learners who learn mathematics using a language that is not their native language. Similarly, Naidoo (2011) describes visualization as the ability to create and transfer the mental images essential for mathematical problem-solving. These mental images, represented through diagrams, pictures, or other visual tools, simplify complex mathematical problems and enhance problem-solving capabilities.

Multilingualism in mathematics education

MLs–those who use three or more languages with varying proficiency (Barwell, 2018; Clarkson, 2016)–bring rich linguistic repertoires to the mathematics classroom. Contemporary scholarship frames multilingualism not as a deficit but as a resource, encompassing social, institutional, and individual competencies across contexts (Franceschini, 2009). These resources can scaffold conceptual understanding in mathematics learning when teachers strategically link learners' home languages with the language of learning and teaching (LoLT) (Moschkovich, 2015; Prediger et al., 2019).

However, the dual demands of acquiring disciplinary content and mastering the LoLT create complexity. Learners whose home language matches the LoLT navigate mathematical discourse more smoothly (Barwell, 2018). In contrast, those whose home language differs must simultaneously decode specialized vocabulary and grapple with abstract concepts (Robertson & Graven, 2018). Multiple-language use can support meaning-making (Moschkovich, 2015), but its effectiveness hinges on deliberate pedagogical design: code-switching risks overloading working memory without clarifying how terminology maps onto mathematical ideas.

Although the socio-political benefits of multilingual approaches are well documented (Planas et al., 2018; Ryan & Parra, 2019), few studies probe how learners' linguistic resources enhance their grasp of mathematical concepts during problem-solving. Existing reviews often foreground language access while sidelining analysis of conceptual learning processes (Moschkovich, 2012; Setati & Barwell, 2006). This gap underscores the need for targeted investigations into how visualization tools can leverage MLs' language repertoires when paired with explicit vocabulary instruction to deepen their conceptual engagement with geometry.

THEORETICAL FRAMEWORK

The commognitive theoretical framework by Sfard (2008) was used to anchor this study. The theory originates from the idea of thinking as a form of communication. According to Sfard (2008), thinking is an individualized form of communication. This emphasizes that one communicates with oneself whilst thinking. She stresses that communication and individual cognitive processes (thinking) are different between thinking and communication.

In its most basic form, commognition is a cohesive theory of "thinking about thinking within oneself." This study adopted two fundamental principles of commognition: thinking as an individualization of communication and mathematics as a form of discourse, which will be explained below.

Thinking as Individualization of Communication

Sfard (2008) asserts that human thinking is a form of communication that happens with oneself that resembles interactive communication. The commognitive view emphasizes thinking as a dialogical process comprising interactions within oneself, such as notifying, arguing, questioning, and responding to thoughts and expressions, whether through verbal or nonverbal communication. This theory proved relevant to geometry problems, as MLs were expected to reflect on their thinking during problem-solving.

Mathematics as a Form of Discourse

Sfard (2007) specified that there are some rules governing both individualized interpersonal communication and interpersonal communication that individuals may follow unconsciously. For this study, during problem-solving questions. Communication is linked to discourse, which unites some individuals and alienates others based on their interests and ability to follow the rules of communication.

Key Commognitive Constructs

Sfard's (2008) commognitive framework identifies four elements of mathematics discourse: word uses, endorsed narratives, routines, and visual mediators. "Word uses" refers to words utilized in mathematics discourses; "endorsed narratives" are sequential patterns that refer to mathematical objects and the relations between these objects, which can be endorsed or rejected within mathematics; "routines" are repetitive patterns, like drawing graphs; and "visual mediators" refer to objects in mathematics, such as diagrams and mathematical symbols (Sfard, 2008). This study engaged with these four constructs in the first and second phases of data collection. As this study focused on ML thinking when solving geometry problems, Van Hiele's (1957) model was employed to determine learners' levels of thinking and understanding during problem-solving – although it was not used as a lens to view the results of the study. While learners' thinking and reasoning levels were not assessed directly in this study, they are relevant in this study as progress from one level to another depends on learners' understanding of all properties of each level–such as word use and visual mediators–during problem-solving

METHODOLOGY

Research Approach and Design

This qualitative study employed an exploratory case study design to investigate how visualization influences geometry problem-solving among MLs. Guided by Fahy's (2016) assertion that qualitative inquiry foregrounds participants' perspectives and experiences, the research focused on MLs' use of visual tools during Euclidean geometry tasks. An exploratory design was chosen because the interaction between multilingualism, visualization, and geometry performance remains under-researched (Bandalos & Finney, 2018). Limiting the context to grade 11 MLs in three secondary schools within the Harry Gwala District ensured a focused, in-depth examination (Zhang & Creswell, 2013).

Selection of Participants

Eighteen grade 11 MLs–whose home language differed from the English LoLT–were selected via purposive convenience sampling across the three schools–all the participants aged below 18 years. Gender was not considered when selecting participants; both males and females were chosen based on the criteria. Researchers designed a qualitative mathematics questionnaire that identified six participants per site, stratified by performance level (two high achievers, two average, and two low performers). This stratification ensured diverse experiences and strategies, enriching the study's insights.

Data Collection Methods

Data were gathered through semi-structured, task-based individual interviews and school-based focus group discussions. Six MLs per school first completed geometry tasks while thinking aloud; immediately afterward, they participated in semi-structured interviews and focus groups to reflect on their visualization strategies and language use while the experience remained fresh.

Data Analysis Procedures

A reflexive thematic analysis, following Braun and Clarke's (2023) six-step framework, was applied to interview and focus group transcripts. Initial codes–such as "diagramming to scaffold reasoning" and "L1 to clarify geometric terminology"–emerged from a close reading of participants' problem-solving sessions. These codes coalesced into broader themes (e.g., "misinterpretation of problem prompts") and were interpreted through commognitive constructs–word use, visual mediators, routines, and narratives–to reveal how MLs' multilingual repertoires and visual strategies jointly shape their geometric reasoning. This integrative approach illuminates both the linguistic obstacles that impede comprehension and the dual cognitive-communicative role of visualization in multilingual geometry learning.

Ethical Considerations

Permission to conduct this study–which builds on a chapter of the first author's PhD research–was granted by the KwaZulu-Natal department of education. Approval was then obtained from the principals of three participating secondary schools. Since all participants were minors, written informed consent was secured from their parents or legal guardians. Ethical clearance was also sought and approved by the University of KwaZulu-Natal's research ethics committee, ensuring that all procedures adhered to the highest ethical standards.

FINDINGS AND DISCUSSION

The study utilized all four key elements of Sfard's (2008) principles of commognition in operation. Three themes emerged from ML's responses to the semi-structured task-based and focus group discussions. The findings present all the key elements simultaneously.

Figure 1 represents the themes from the data generated using semi-structured task-based interviews, semi-structured interviews, and focus group discussions.

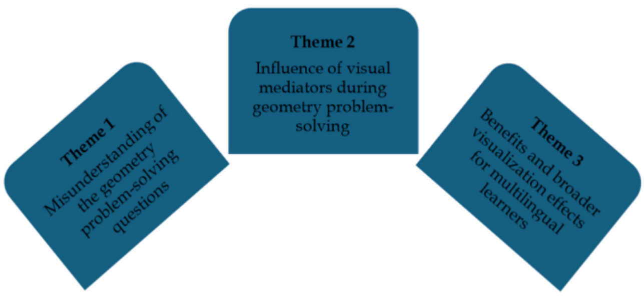
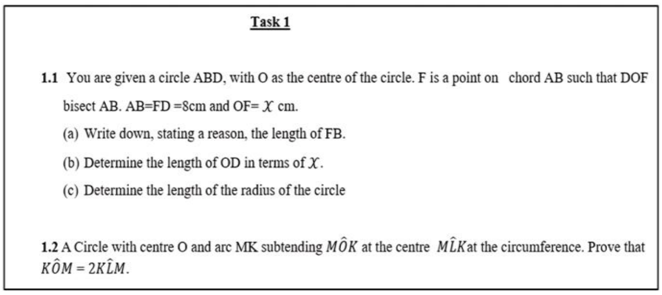


Figure 1. Themes identified during data analysis (Source: Authors' own elaboration)



Task 1

1.1 You are given a circle ABD, with O as the centre of the circle. F is a point on chord AB such that DOF bisect AB. AB=FD =8cm and OF= X cm.

(a) Write down, stating a reason, the length of FB.

(b) Determine the length of OD in terms of X.

(c) Determine the length of the radius of the circle

1.2 A Circle with centre O and arc MK subtending $M\hat{O}K$ at the centre $M\hat{L}K$ at the circumference. Prove that $K\hat{O}M = 2K\hat{L}M$.

Figure 2. Semi-structured individual task-based interview (Source: Research instrument)

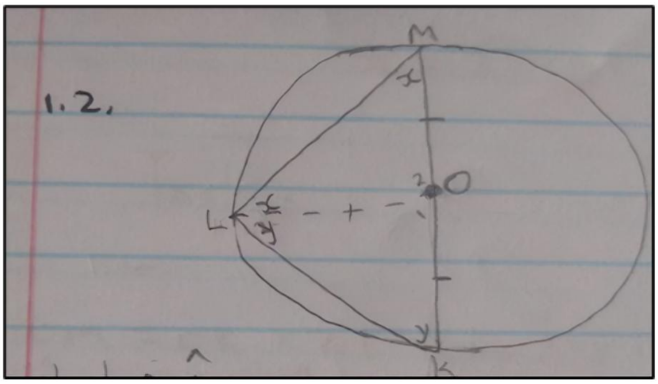


Figure 3. A5 response to task 1 (Source: Field study)

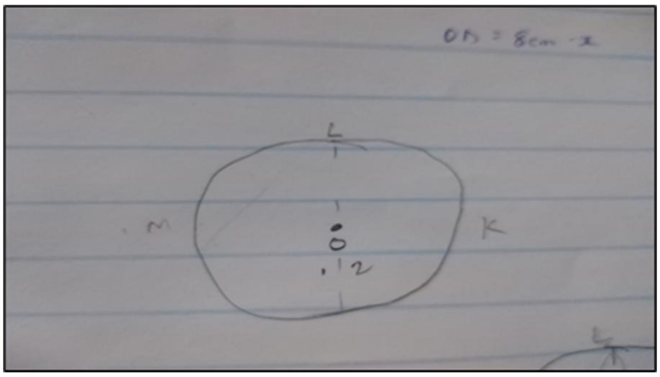


Figure 4. C3 visual mediator for question 1.2 of task 1 (Source: Field study)

Misunderstanding of the Problem-Solving Questions

This section presents and analyses participants' misunderstandings related to word usage when interpreting the problem statement. It then examines the types of visuals participants used to aid their understanding of the problem statement, providing insights into their comprehension strategies in task 1. Task 1 of the task-based interview is shown in **Figure 2**. The focus of task 1 was on the visualization used by participants when no diagram was provided. How they solved the task was equally crucial to analyzing their understanding of the statement before solving the problem. The focus was not on finding the correct solution but on how language (word use) influenced their understanding of the problem.

A5 and C3 used their understanding of concepts to draw a diagram to help them visualize the problem before solving it. However, the diagrams (**Figure 3** & **Figure 4**) reveal that the participants did not understand the geometry concepts or words used in the tasks, as they did not represent these concepts accurately in the diagram. According to Hasanah et al. (2019), teaching mathematics aims to equip learners with thinking and problem-solving skills and develop their ability to communicate mathematical ideas or convey information through diagrams. This view suggests that these participants have not met the objectives of teaching mathematics regarding their understanding of geometry concepts. Based on the participants' diagrams, their failure to understand the geometry terminology such as arc, subtending, chord, and bisect used in the problem statement to alert them hampered their ability to visualize the statement precisely. As a result, they failed to execute their procedure or routine of solving the problem using the diagram. This poor understanding was confirmed by the participants during focus group discussions; the participants shared their experiences in understanding the problem statement:

> **A5:** I thought I must draw a diagram after reading the statement, and I did, but when I was trying to respond to questions, it was difficult for me to answer the questions …

> **C3:** I was challenged by the name "arc" in 1.2. I did not understand what it meant, and then I failed to move on to answer the question.

Similarly, **Figure 5**, **Figure 6**, and **Figure 7** represent visual mediators used to understand the problem.

C3, B6, and B4 used their understanding of concepts to draw a diagram to help them visualize the problem before solving (**Figure 5**, **Figure 6**, and **Figure 7**) and concerning Yin (2010), five processes and seven roles of visualization adopted in this study–which are understanding, connecting, constructing, using the visual representation to solve the problem, and encoding to answer the problem–the participants failed to understand the statement, which contributed to their failure to move to the following process–connection–and use the visual representation to solve the problem. One may be unable to communicate about a geometry concept with oneself or others if the concept cannot be seen or imagined (Sfard, 2008). Objects are seen to "help interlocutors in making discursive decisions and sustaining these sense of mutual understanding" (Sfard, 2008, p. 147). Sfard (2008) underscores the importance of helping interlocutors make thoughtful, well-reasoned decisions while maintaining a shared understanding in their discussions. This view was not seen in the discursive actions of the participants: their diagrams were inaccurate, and they failed to engage in communication using geometry concepts; hence, they

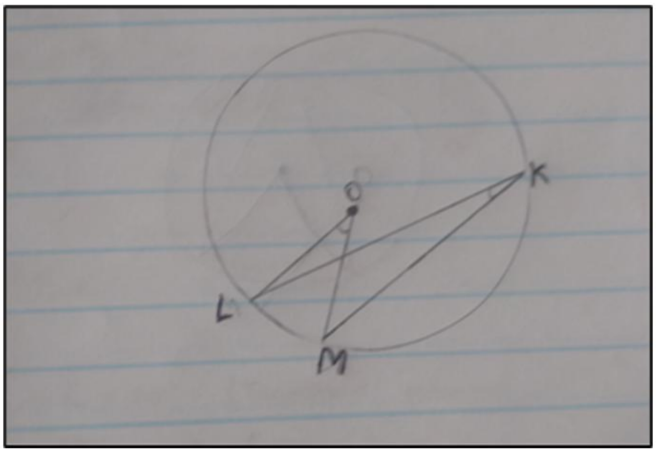


Figure 5. C3 response to 1.2 of task 1 (Source: Field study)

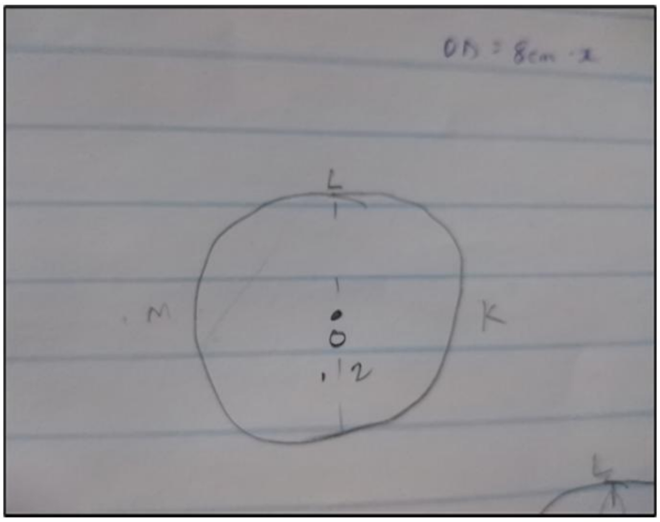


Figure 6. B6 representation of task 1 (Source: Field study)

failed in the first role of visualizing by Yin (2010). As a result, they did not come up with any other correct routines to solve the problem, and the absence of a narrative confirmed a lack of understanding of the problem based on their communication with themselves during problem-solving (**Figure 5**, **Figure 6**, and **Figure 7**).

Given that these learners were in their eleventh of schooling, their difficulties indicate that while they may have encountered geometric concepts before, they might not have developed strong visualization skills or the ability to translate verbal descriptions into accurate diagrams (**Figure 3**, **Figure 4**, **Figure 5**, **Figure 6**, and **Figure 7**). This challenge is further compounded by language barriers, as MLs often face difficulties in interpreting mathematical terminology and problem statements, especially when the language of instruction differs from their home language. Research suggests that language plays a crucial role in mathematical understanding, as it influences how learners process, interpret, and apply mathematical concepts (Sfard, 2008). Consequently, their inability to construct appropriate diagrams stems from a lack of exposure to effective visualization strategies and linguistic challenges that hinder mathematical comprehension and problem-solving. In terms of Van Hiele's (1957) geometric levels of thinking, as these participants could not produce a diagram using written information or depict a geometric relationship such as a theorem in a diagram, they were found to be operating at Level 2 in their geometry thinking. As a result, they failed to progress to the next step of devising a proper plan(routine) to solve the task.

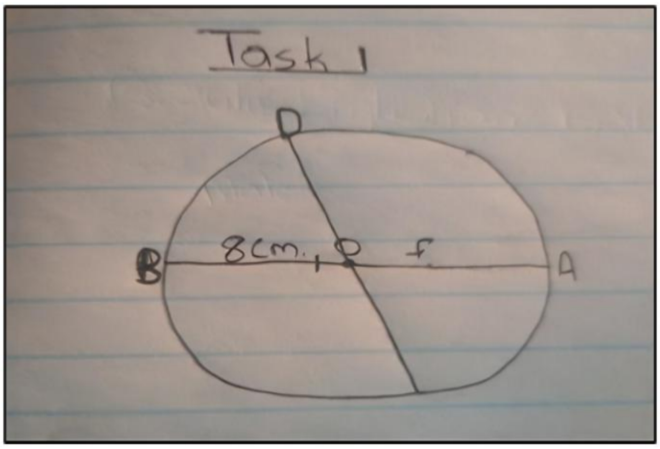


Figure 7. Response of participant B4 for task 1.1 (a) (Source: Field study)

The findings suggest that the few MLs who attempted to draw diagrams (e.g., B6, A4, B4, A4, A5, and C3) relied heavily on visual representation to understand geometry concepts. They believed that diagrams played a crucial role in solving problems, as they struggled to explain their reasoning using statements alone. Instead, they used sketches to support their thinking. Mudaly and Reddy (2016) highlight that visual representation aids learning by facilitating reflection and communication of mathematical ideas, while Sfard (2008) notes that mathematicians often use visual imagery in abstract reasoning. The participants' reliance on diagrams aligns with these perspectives, indicating that visualization helped them better comprehend and approach problem-solving in geometry.

During the focus group discussion, the participants explained that:

> **B6:** For task 1, since there was no diagram to refer to, it was difficult for me to respond to the question using the statement only ...
>
> **B4:** For me, task 1 was difficult …
>
> **C3:** I did not understand some words in the statement. As a result, I failed to draw a diagram.

The participants' responses and their visual mediators shown in **Figure 3**, **Figure 4**, **Figure 5**, **Figure 6**, and **Figure 7** indicate that most wanted to begin solving the problem by drawing a diagram to visualize the problem; however, because they did not understand the statement, they were unable to draw a diagram or use the key geometric properties from the problem statement to reach a sound understanding. The participants' accounts highlight that keywords or geometry terminologies are vital in understanding geometry problems. They found it challenging to understand the question because they lacked an understanding of geometric vocabulary such as 'arc' diameter and bisect used in the problem statement. The geometry terminology(words) that are used when learning about circles contributes to a distinct discourse (Sfard, 2008). In addition, Atebe and Schäfer (2010a, 2010b) assert that language (term use and terminologies) is an essential tool in communication to visualize the problem and identify key elements to solve the question. This affirms that MLs must be able to understand geometry concepts very well to visualize the issues appropriately. Hence, it is evident that language barriers hampered the participants' ability to bridge between understanding and visualization. The participants found translating the problem statement into mental representation challenging, which is critical for problem-solving.

Although most participants encountered problems completing task 1, a few completed it easily. The comments of participants A1 and A2 on their interpretation of word use for task 1 exemplify this:

> **A1:** For me, it was easy in task 1; I was given a statement only, then I read the statement with understanding, then after I created a picture on my mind and drew a diagram, everything was fine, then I calculated the questions.

> **A2:** The statement they said was that the circle I drew started drawing chord AB, and they told me that DOF is bisecting AB. I put 90° as it was said they bisect, then after that, I remembered the theorem that says "line drawn from the center of the chord to the midpoint of the chord bisect the chord." Then after I knew that $AB = FB$.

A1 and A2 narratives agreed with Mudaly and Rampersad's (2010) concept of visualization processes rooted within internalization and externalization activity. **Figure 8** and **Figure 9** demonstrate their visual presentation.

These participants (A1 and A2) demonstrated a good understanding of the terminology (words) used in geometry tasks and applied strategies to respond to questions. This resonates with the emphasis of Sfard's (2008) commognitive framework that keywords play a significant role in mathematics discourse. Sfard (2008) infers that the development of words, which are associated with objects of discourse, forms the basic building block in the phase-driven stage of learning. These words are not only linked to objects but are also

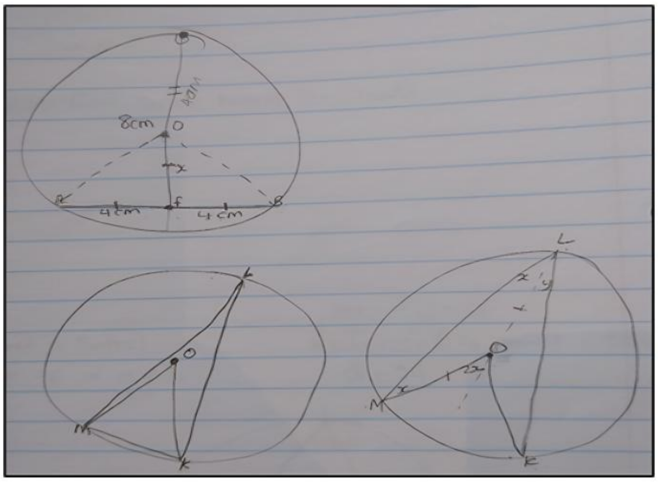


Figure 8. A1 visual mediator for tasks 1.1 and 1.2) (Source: Field study)

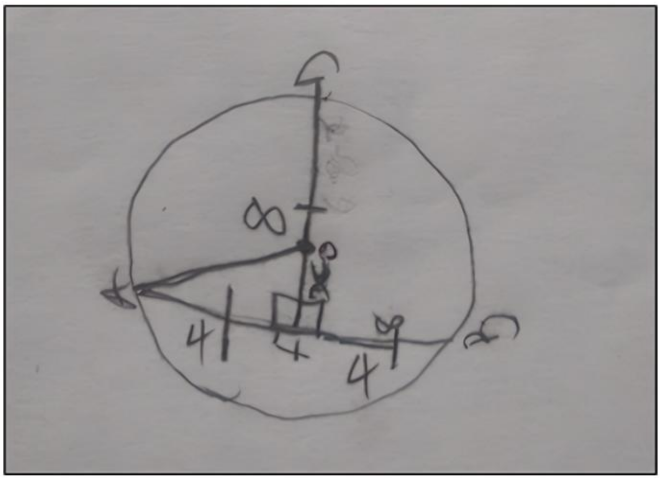


Figure 9. A2 visual mediator for task 1.1 (Source: Field study)

integral to the communicative actions and cognitive processes involved in the construction of meaning through discourse, reflecting the interplay between language and thinking in the process of learning. Learners at this stage can describe geometric shapes with understanding using more flexible words, and they can provide an accurate description of geometric figures.

In relation to Van Hiele's (1957) geometric thinking levels, A1 and A2 were operating at level 2, where they begin to understand the relationships between the properties of geometric figures from the statement and can demonstrate drawing a correct diagram based on the problem statement. Participants A1 and A2 showed that they were performing at level 3 and level 4, where their thinking and reasoning (narratives) were engaged with understanding the meaning of deduction. Hence, they were able to produce a diagram from written information and were able to use deduction to prove a theorem. Lastly, they could use geometry properties accurately, which informed their correct routine, which was validated by endorsed narratives to demonstrate a good understanding of geometric concepts as

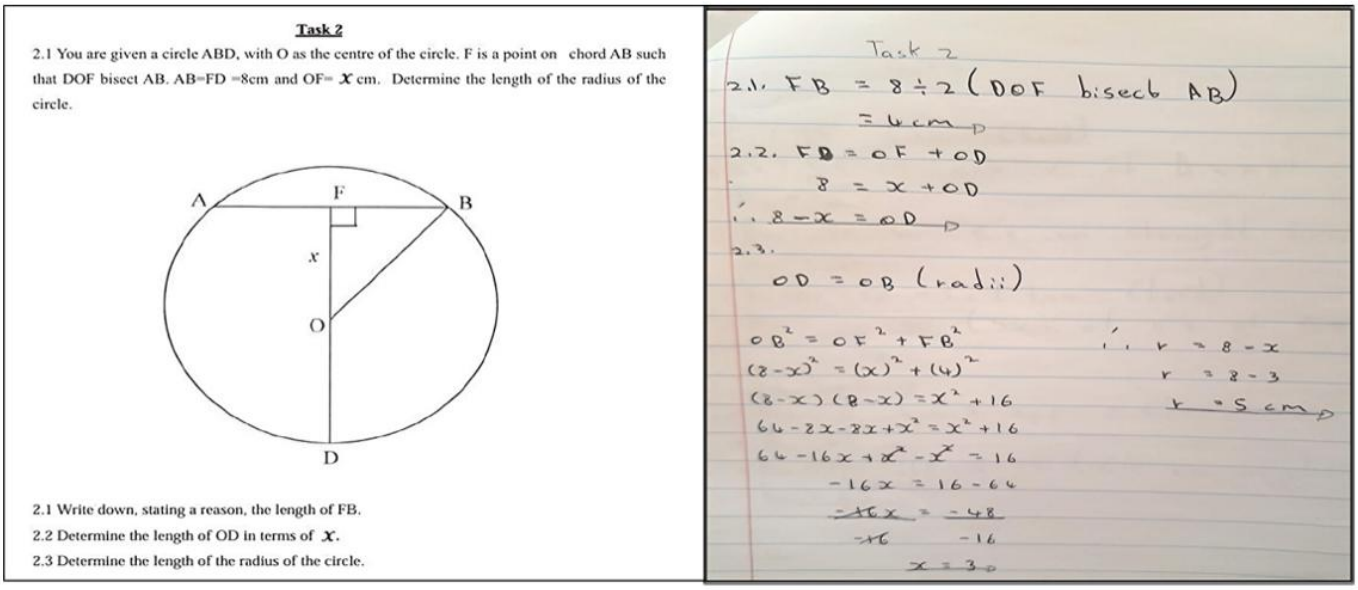


Figure 10. A5 solution to task 2 (Source: Research instrument and field study)

demonstrated in their visual demonstrations (**Figure 7**, **Figure 8**, and **Figure 9**). Concerning Pólya's (1945) steps of problem-solving, A1 and A2 demonstrated all the steps (routines), and the result was that they arrived at correct solutions. This confirms that understanding the mathematical concepts enabled these learners to think of different problem-solving methods.

Influence of Visual Mediators During Geometry Problem-Solving

This section presents a discussion and analysis of the influence of visuals on participants' understanding of tasks 2, 3, and 4 and their comments during semi-structured interviews and focus group discussions to enable further insight into their thought processes regarding the influence of visuals–in this case, diagrams. The analysis of the impact of visuals on their routines focuses on their use of ritual and explorative discourse in the strategies they used to solve the tasks.

A5 In every step of her algorithms(routine), she substantiated using endorsed narratives better than in task 1 (**Figure 10**). Her justification involved the endorsed narrative of "radius bisects the chord." In their case study, González et al. (2021) categorize this endorsed narrative as a verification of sufficient conditions routine. This finding reveals that, for learners who experience challenges in understanding geometry concepts, visuals such as diagrams shape their visual thinking in guiding their critical thinking during geometry problem-solving questions, especially in multilingual contexts where learners experience language barriers that hinder their ability to comprehend geometry concepts as it was evident in this study (**Figure 3**, **Figure 4**, and **Figure 5**).

In addition, the results align with those of Supardi et al. (2021), who state that visual mediators in the form of images or diagrams can aid learners in solving mathematical problems, thus reducing the likelihood of errors. Various narratives, such as line bisecting and radii, were observed under the influence of visuals. In **Figure 10**, A5 engaged fully in Pólya's (1945) steps of problem-solving when visuals were attached to the same problem statement; she failed to understand without any visuals in task 1 (**Figure 3**).

The results of B1's solution resonate with Mudaly and Reddy (2016) study about the role of visualization in proving the processes of Euclidean geometry, which emphasizes that diagrams play a significant role in proving geometry processes. Vale and Barbosa (2018) found that using strategies that require visual representations may facilitate problem-solving strategies. This was evident in the present study, as participant B1 manifested a good understanding of problems when accompanied by diagrams (**Figure 11**). This study shows that visuals play a significant role in learners' knowledge of the problem in multilingual settings, as fewer language errors were identified in participants' responses to task 2 and task 3 than in task 1. This view concurs with the literature review that found that visuals were compelling when teaching learners in English whose first language was not English (Bautista & Ortega-Ruiz, 2015; Carrier, 2005; Harbi, 2024; Maries & Sing, 2013).

The data demonstrates that the participants substantiated their solutions concerning the statement and the diagrams using a set of rules when solving the tasks (**Figure 8** and **Figure 9**). Using a strict set of rules when solving the task is a characteristic of the ritualistic phase (Sfard, 2008), which later enables learners to develop their explorative ability to solve mathematics problems. Sfard (2008) maintains that mathematics education aims to develop learners' explorative ways of

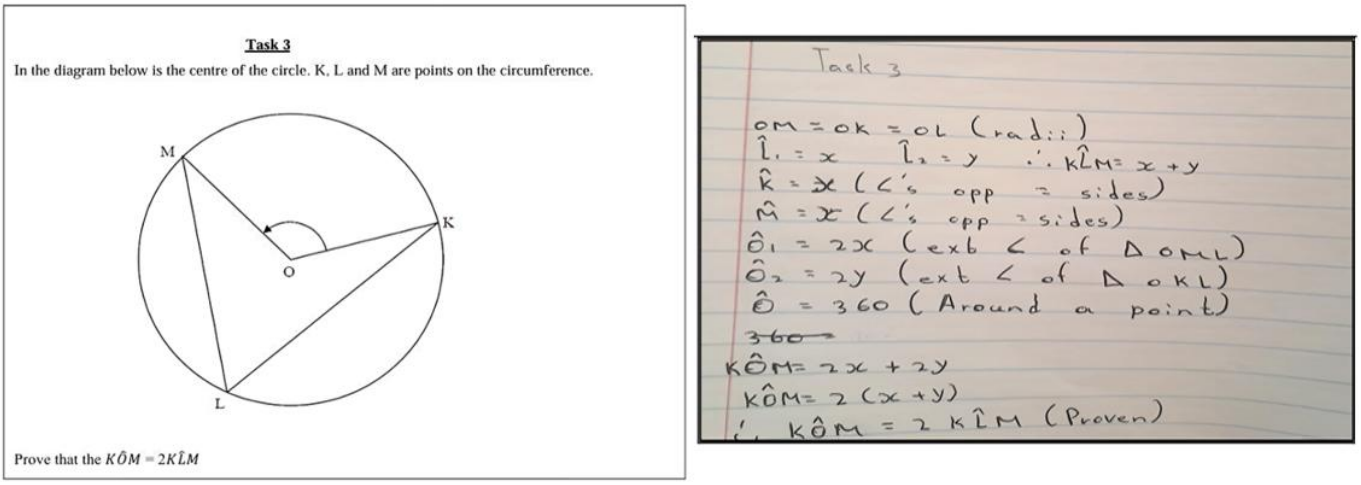


Figure 11. B1 solution of task 3 (Source: Research instrument and field study)

reasoning. When participants explain during focus group discussions regarding the presence of diagrams:

> **A2:** The diagram played a significant role. I was able to figure out how I would be able to find the angles required in terms of their position.

Most of the participants across the three sampled schools validated their understanding of geometry concepts in relation to the diagrams and knew the purpose of the diagram (**Figure 10** and **Figure 11**). These findings are supported by previous literature that states that diagrams are significant and, in most cases, are key to solving geometry problems (Yahya et al., 2022). Yahya et al.'s (2022) view was validated by the participants; for instance, C1 explains that:

> **C1:** Diagrams were more helpful to me because, for example, when I was solving task 4 when I was calculating angles on a straight line like $\hat{B}_2$ when I was done, I had to add $\hat{B}_1$ and $\hat{B}_2$ they must give me 180°.

Klerlein and Hervey (2020) conclude that exposure of learners to strategies that enhance their understanding helps to equate learners with the range of strategies required to solve a problem and understand that some tasks may be solved by applying more than one strategy. This was evident in this study in the questions with diagrams: participants substantiated multiple strategies for solving the task–especially for task 4 (**Figure 12**). In South African contexts, where most schools lack resources, and most learners are learning in a language other than their home language, the findings suggest that visuals provide a powerful tool to overcome language barriers experienced by participants. This finding was evident in most of the participants like A5, the participant experienced challenges in comprehending and visualizing the task 1 problem statement at the same time without any visual (**Figure 3**) however when the diagrams were provided (**Figure 10**),

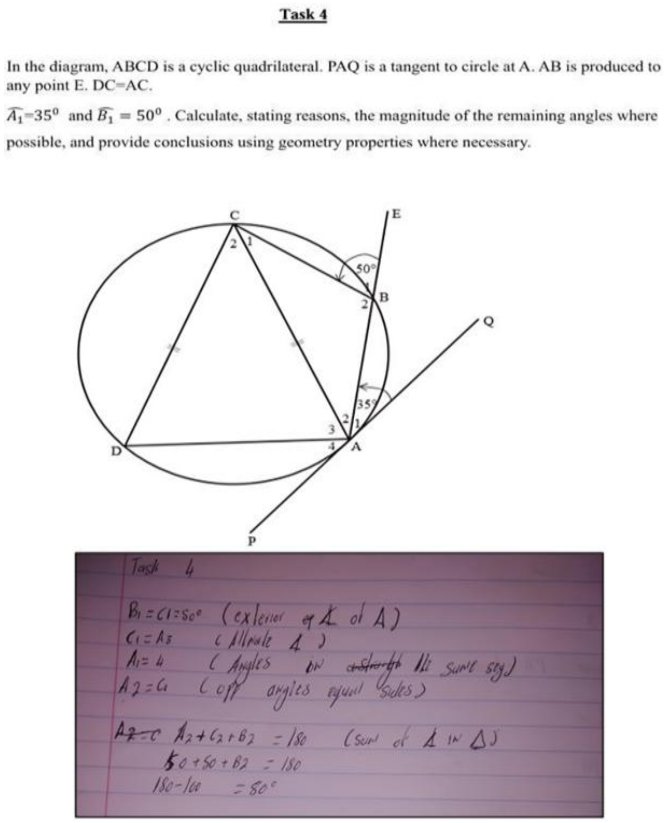


Figure 12. B2 task 4 solution (Source: Field study)

the participant managed to understand the problem statement and integrated it with the visual mediator (diagram) which demonstrated a good understanding of the problem statement. This was confirmed by applying a correct routine and executing well, providing endorsed narratives of her actions (**Figure 10**). This finding correlates with a study by Moleko (2021) that found, in a multilingual context, that diagrams provided learners with various options during problem-solving and enabled them to represent more information using visual representations, reducing the language barrier.

Some tasks–such as task 1.2 and task 3–required ritualized practices. Some participants documented their practices during problem-solving. For example, for task 4, participant B2 (**Figure 12**) documented their practices as follows.

Task 4's diagram was designed without parallel lines, without angles on the same segment, and with a transversal line to demonstrate alternating angles. B2 applied the 'equality of the angle' property associated with parallel lines (**Figure 12**). The participant decided that the lines were parallel, although nothing in the diagram or the statement communicated that the two lines were parallel. B2 thus demonstrated ritualistic thinking by over-generalizing the properties of parallel lines to any task with a similar appearance. While, in appearance, the lines looked parallel, in geometry, properties must be communicated either through a statement (word use) or with a visual mediator, such as a symbolic or iconic mediator, and must not be assumed based on appearance. The participant failed to analyze the diagram appropriately as the second step after recognizing the appearance for assurance.

Sfard (2008) also affirms that signifiers are used in geometry discourse to communicate or describe certain essential features of the task to the learner. B2 applied parallel lines inaccurately and thus arrived at a wrong answer ($\hat{A}_3 = \hat{c}_3$) and provided a wrong narrative to substantiate his answer–using 'alternating angles.' By looking at the diagram, B2's response demonstrates that he knew how alternating angles are positioned but did not know when the parallel lines property may be applied. His line of thinking prioritized the visual appearance of the lines on the diagram rather than identifying a signifier or iconic mediator that would indicate the relationship between the lines.

This finding demonstrates that diagrams are significant for MLs when solving geometry questions, but learners must focus on signifiers in the diagram and not rely on the appearance of the diagrams. Learners are expected to analyze the diagrams and all the information they communicate. This finding resonates with previous research by Supardi et al. (2021) that found that learners may still make mistakes despite using correct visual mediators. Another study by Rellensmann et al. (2017) revealed that using appropriate visual mediators did not guarantee accurate solutions. This demonstrates that, in multilingual environments, learners may also not know the proper problem-solving strategy even when a diagram is provided. This is due to the inability of learners who are not fluent in the required mathematical register and vocabulary to integrate geometry concepts and visuals. Not understanding the mathematical language and register results in problem-solving difficulties, thus causing mathematical operations misapplication (Moschkovich, 2002).

Similarly, A6 and B6 experienced more challenges, especially for tasks 1, 2, and 3. They tried to attempt task 4. This finding demonstrates that task 4 took their

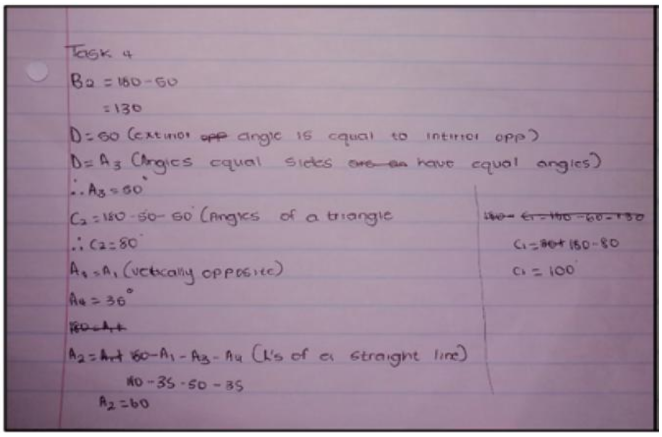


Figure 13. A6 task 4 solution (Source: Field study)

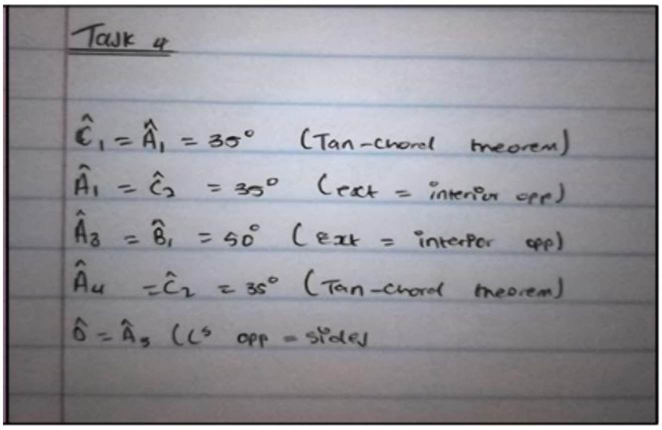


Figure 14. B6 task 4 solution (Source: Field study)

attention as more information was presented using visuals, **Figure 13** and **Figure 14**.

The participants paid significant attention to some essential features of task 4; however, they applied some of the properties of shapes incorrectly. In A6's solution, there are many cancellations, which displays uncertainty in how she planned to solve the task (**Figure 13**). This was demonstrated at the start of her solution: she managed to get the size of $\hat{B}_2$, but provided no narrative to support the solution. This confirmed that she paid more attention to visual representation than to the narrative. In addition, when solving the angles, she used incorrect narratives, like $\hat{c}_2$. Both participants seemed to not pay attention to the 'why' in relation to commognition, as discussed by Sfard (2008). A6 misapplied some geometry properties; for example, she said $\hat{A}_4 = \hat{A}_1$ as vertically opposite angles, which resulted in her arriving at the wrong values for these angles (**Figure 13**). In addition, she could not use the diagram to verify the sum of the angles, like some angles of a triangle must add up to 180°. Diagrams are significant in ascertaining solutions in geometry (Mudaly, 2016).

Similarly, B6, in task 4, argued that $\hat{A}_3 = \hat{B}_1$. She substantiated this by saying the 'exterior angle is equal to the interior opposite angle' (**Figure 14**). She seemed to misunderstand the concept of cyclic quadrilateral angles or was unsure whether it was the exterior of a triangle or the exterior of a cyclic quadrilateral. Both participants lacked concentration in using geometric properties when solving geometry questions, resulting in some incorrect answers. These wrong assumptions have resulted from the participants' internalization. They seemed to have weak iteration between their visual and thinking processes (Mudaly, 2021), which resulted in wrong answers. It appeared that they could demonstrate their active participation in geometry questions when diagrams were provided compared to problem statements alone (see **Figure 6**). Still, they needed to attach their thinking to the visuals and information provided in the diagrams. This finding demonstrates that they lacked an understanding of these geometry concepts in relation to diagrams; they were not proficient in understanding these concepts and, as a result, they only relied on the appearance of the diagrams, which did not communicate the geometric properties they were using in supporting their narratives. Hence, there were no geometric signifiers they referred to like parallel lines.

During focus group discussions, B6 responded that:

> **B6:** I cannot respond to questions if I must use English …

These participants seemed to experience a high cognitive load due to language barriers to the comprehension of geometry concepts and visualization. Strong language skills in a learner's first language enhance problem-solving abilities for problems that require an understanding of complex linguistic structures (Peng et al., 2020).

Benefits and Broader Visualization Effects for Multilingual Learners: Problem-Solving Geometric Discourse Analysis

This section presents the data generated using semi-structured interviews, semi-structured task-based interviews, and focus group discussions. The chapter commences by presenting participants' discourses during problem-solving, followed by an analysis of the benefits of visualization to participants' understanding.

Participants articulated their understanding of the tasks as follows.

> **A2:** In task 3, I was required to prove the theorem that says the angle at the center is two times the angle at the circumference. I applied the knowledge I got from my teacher to prove the theorem. In task 4, I found it very easy because, in the statement, they told me that ABCD is a cyclic quadrilateral, and then I applied all the knowledge about cyclic quadrilateral.

A2 demonstrated a good understanding of the questions, particularly when diagrams were presented; her visualization ability was excellent (**Figure 15**). Her

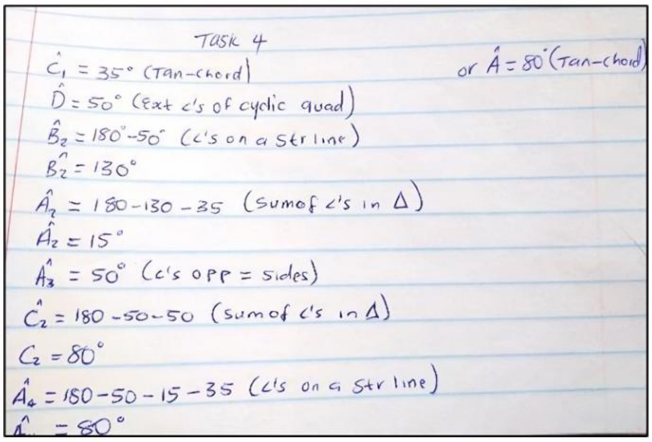


Figure 15. A2 task 4 solution (Source: Field study)

understanding appeared to be attached more to geometry theorems, rules, and concepts. She highlighted the importance of what was provided in the statement and diagram in task 4. In the diagram, there were other properties that the participant should have highlighted. However, she thought the ones she mentioned were adequate. In analyzing her explanations, she seemed to visualize better when the diagram was provided in the question, as she started with task 3. For further clarity, she was requested to indicate the difficulty of the tasks, from easiest to most difficult. She stated:

> **A2:** 'The easiest was task 4, followed by task 3, followed by task 2, and, lastly, task 1'.

Figure 15 represents A2 response to task 4.

Participant A2 demonstrated that the diagrams presented were meaningful and that her previous knowledge enhanced her cognitive processes for visualizing during problem-solving. Pi et al. (2023) support the view that an absence of prior knowledge and a lack of construction of mental ideas result in learners encountering challenges in visualizing and assessing their thinking.

A1 also expressed her understanding of the questions.

> **A1:** In task 4, in the statement, I was told that ABCD is acyclic quadrilateral; I thought of all the properties my teacher said to me regarding the cyclic quadrilateral, also the ones I discovered on my own; secondly, PAQ is a tangent. I also remembered all the tangents' properties. Then, after I was told that DC = AC, I went to the diagram to confirm that. After that, I went to all angles that were given, starting with $\hat{A}_2$, I was told by my teacher that when I see the tangent, I have to my fingers at the end of the chord and drag my fingers they will take me to the angle equal to the angle at the alternate segment, which resulted to $\hat{A}_2$ being equal to $\hat{c}_1$ …

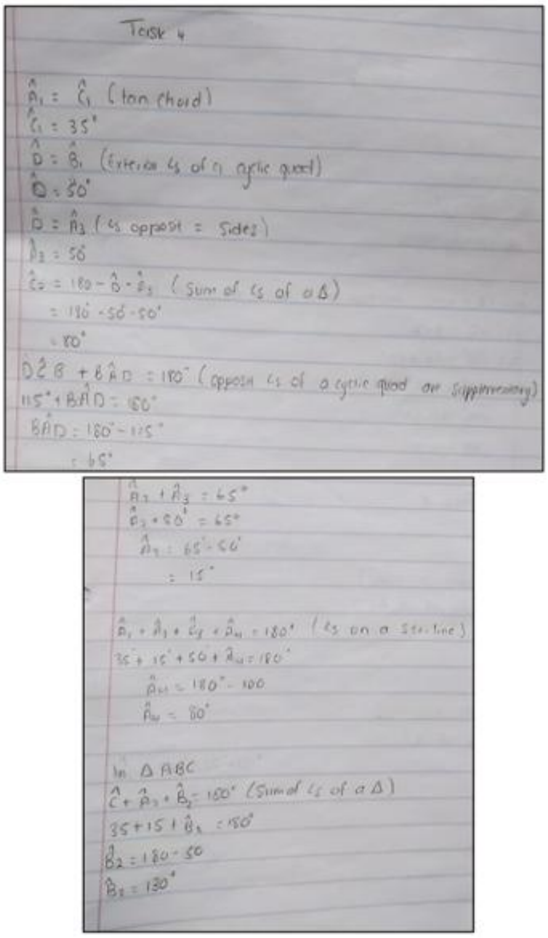


Figure 16. Participant A1's response to task 4 (Source: Field study)

Figure 16 represents A2 response to task 4.

Analysis of A1 and A2 responses demonstrates the importance of visualizing in geometry. A1 and A2 remembered important geometric properties, such as theorem reasoning, geometry shapes, and the importance of signifiers that are included when visualizing geometry questions using diagrams (**Figure 15** and **Figure 16**). However, the two participants started with task 4 rather than task 1. As task 1 provided a statement only, with no diagram, this emphasizes that they found the questions that were presented using diagrams easier to answer (**Figure 2**). This affirms that visualizing geometry concepts using diagrams played a significant role in these MLs' understanding of the questions and enhanced their confidence in explaining geometry concepts. Vale et al. (2018) agree that the use of visual methods during mathematical problem-solving is vital, as, without them, the chances are greater that learners will attempt to solve problems without adequate understanding.

For task 1, A2 answered correctly (**Figure 9**), while A1 (**Figure 8**) used the correct method. She had difficulty at the end determining the length of the circle's radius. Task 4 was the only question they could confidently answer and justify their answers.

The findings show that, in this multilingual context, visualization played a significant role in shaping and influencing learners' understanding of geometry problems. Visualization seems to benefit MLs more as they are not fluent in the LoLT. This finding resonates with those of studies by Barwell et al. (2007) and Robertson and Graven (2020) that established that in multilingual settings, learners must deal with the additional challenges resulting from not being fluent in the LoLT. Also, Samosa et al. (2021) conducted a study about how effectively it is to visualize, represent, and solve problem techniques to improve learners' problem-solving skills. The findings agree that visualization techniques also assist learners in understanding a problem, exploring possible solutions, and developing new perceptions. For these MLs, visualization was found to help them to be explorative in their thinking during geometry problem-solving questions. In global research, MLs often exhibit similar patterns, with visual aids helping to reduce cognitive strain. This was evident from studies in countries like India and the U.S., where visual supports have proven beneficial for both multilingual and monolingual learners in complex subjects like geometry (Presmeg, 2006). Hence, this issue is not particular to South Africa alone.

South Africa is, perhaps, an extreme example of a multilingual society in which a single, non-indigenous language dominates. English is South Africa's primary language of teaching and learning. While census data indicates that it is the home language of less than 10% of the population, South Africa's department of basic education reports that learners are officially learning in and through English (Department of Basic Education [DBE], 2014). The majority of South African learners have limited access to their native language as a resource for understanding geometry concepts in their classrooms. This is most evident in rural contexts where learners have to learn mathematics using English, which adds more complexity to their understanding of geometry concepts. This necessitates finding more effective ways of enhancing South African MLs' knowledge of geometry concepts rather than relying on LoLT. Hence, from the study's findings, visualization was found significant for learners who learn geometry using a language that is not their home language.

In addition, C5 articulated her understanding of the tasks as follows:

> **C5:** In task 4, I was told that POA is a tangent, then, after I remembered all the theorems associated with the tangent, however, I realized that only one theorem is applicable, the tan-chord theorem …

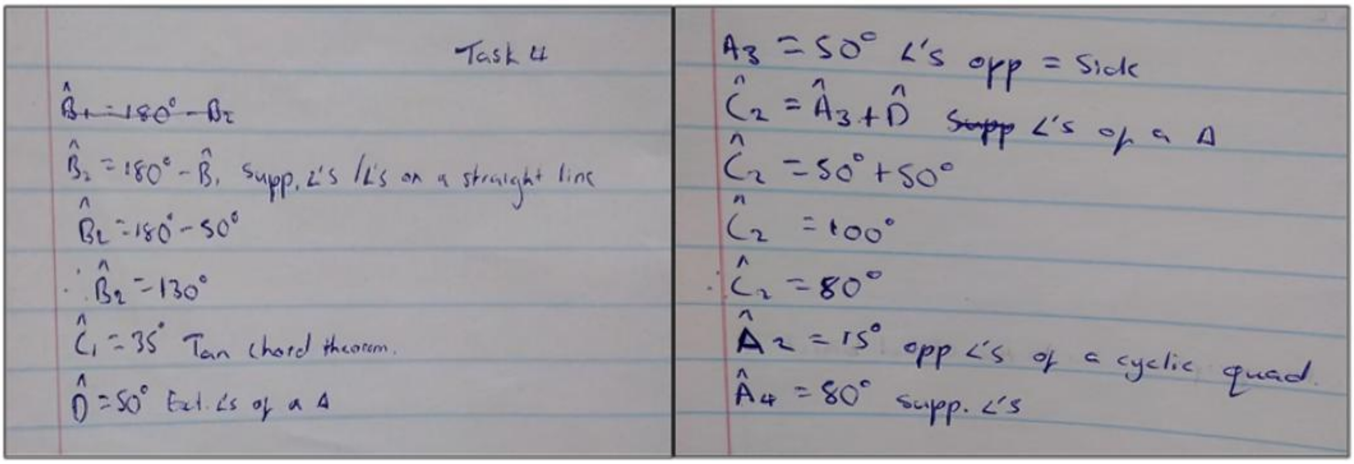


Figure 17. C5 solution to task 4 (Source: Field study)

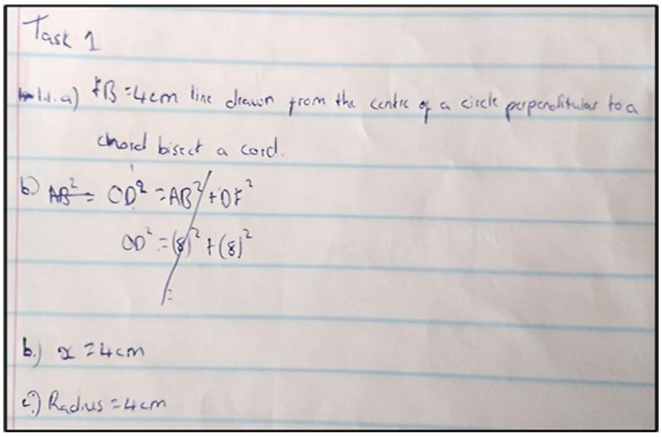


Figure 18. C5's solution to task 1 (Source: Field study)

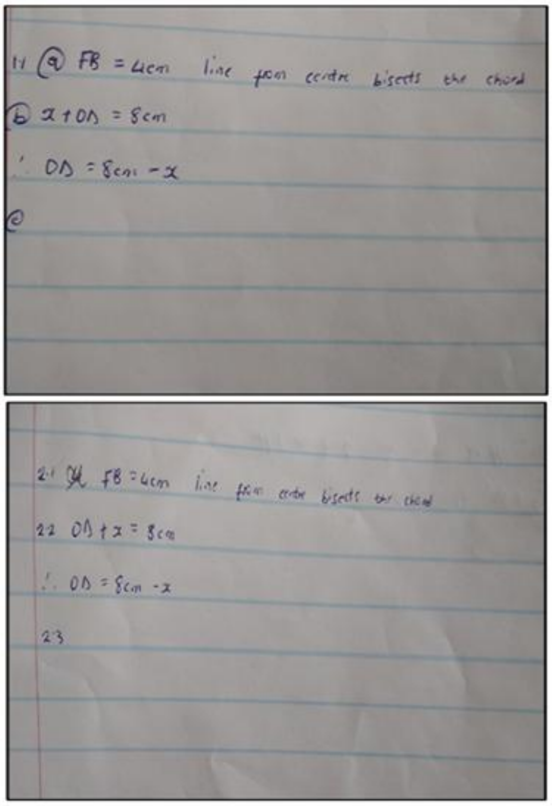


Figure 19. C3's solution to task 1.1. (a) (Source: Field study)

C5's understanding was also evident in her solution regarding the algorithms she used and her interpretation of geometric shapes, theorems, and iconic mediators (**Figure 17**). This finding demonstrates that seeing was crucial for her visual abilities, hence she experienced challenges of understanding problem statements in task 1 (**Figure 18**).

Furthermore, C3 shared her experiences in understanding the questions:

> **C3:** Diagram provided a better understanding of the questions; usually, I forget when I am given a statement only, so I have to read the statement several times; in this case, I was given a diagram, and it was easy for me to understand, for instance, in task 4, in the statement it was not specified that ABC is a triangle, but as the diagram given, I was able to see that on my own.

For task 1, C3 only answered task 1.1 (a) correctly and provided an endorsed narrative, as shown in **Figure 19**.

C3's solution for task 4 is shown in **Figure 20**.

Both C5 and C3 performed poorly on task 1 (**Figure 18** & **Figure 19**). However, when the diagram was given, they could see the geometric properties of the problem and enhanced their understanding of the problem (**Figure 17** & **Figure 20**). This finding was also evident in A5 solutions for task 1 and task 2 in **Figure 3** and **Figure 9**, where she solved the problem well only when the diagram was provided. This finding validated that visualization is the key to solving geometry questions for MLs who experience the challenges of mathematical vocabulary.

In addition, B1 shared his experiences of the tasks:

> **B1:** Task 1, for me, was a bit challenging because I believed that I needed a diagram to guide me; however, I couldn't manage to draw it–especially for 1.1. In conclusion, the key was that you must understand the statement without a diagram. For

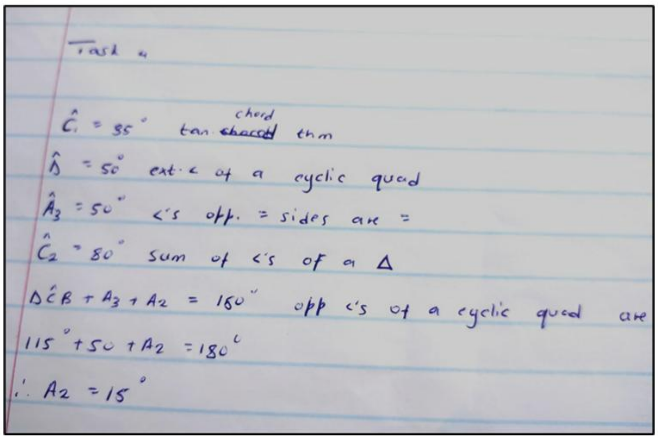


Figure 20. Participant C3's use of visual mediators in their solution to task 4 (Source: Field study)

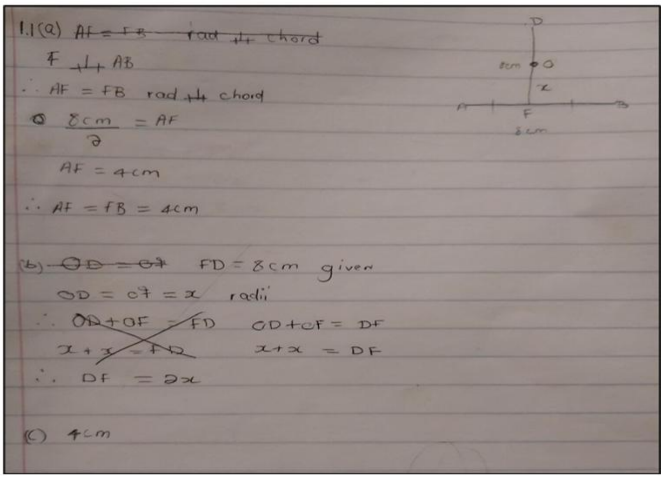


Figure 21. B1 solution to task 1.1 (Source: Field study)

> instance, the question about the length of a radius was more problematic to me. Thus, I did not understand the statement well when I realized that even my diagram was wrong.

The B1 challenge was also evident in his solution to task 1. **Figure 21** demonstrates his solution to task 1.1.

In the B1 solution (**Figure 21**), there are a lot of cancellations with narratives that are not mathematically accurate, which revealed that word problem statements challenged him to understand. Furthermore, this was evident in his visual mediator. B1 did not attempt to answer 1.2, which was the same as task 3, and stopped at 1.1 (c), which was incorrect. However, in task 3, **Figure 11**, B1 demonstrated a good understanding of the problem when accompanied by diagrams, answering the question accurately compared to task 1, without any cancellations. This finding reveals that, in multilingual contexts, learners experience language barriers in trying to comprehend geometry concepts alone, which hinder their ability to engage fully in understanding geometry problem-solving processes. However, visualizing geometry problems in this study using diagrams was found necessary in bridging language barriers and enhancing ML's understanding of the problem.

These findings suggest that most MLs benefit from a visual presentation of geometry concepts (**Figure 14**, **Figure 15**, **Figure 16**, and **Figure 17**). Visuals enabled participants better to understand the geometry concepts during the problem-solving process, as it was easier to communicate their understanding using diagrams. Some previous research also reported that learners benefit from visual and verbal representations of mathematical concepts (Jones, 2013; Mudaly, 2012; Sfard, 2008). In a multilingual context, it is evident that the use of visuals is effective as it eliminates language as a factor by providing a universal mode of communication for understanding geometry concepts during problem-solving (Chval et al., 2020). These findings are supported by Mudaly and Naidoo (2015), who states that for mathematics instruction to be beneficial and appropriate, given the various learning styles of learners, concepts need to be presented using multiple visual instructional strategies. In this study, such visuals enhanced MLs' ability to understand the meaning of the tasks. In rural schools in South Africa, where learners may not have equal proficiency in the language of instruction and the mathematical register, visualization can provide a universal mode of comprehension (Chval et al., 2020). These findings were also supported by previous research by Planas and Civil (2013) and Prediger et al. (2018) that emphasized that MLs benefit from multimodal approaches–especially visuals–in learning mathematics.

CONCLUSIONS

The findings underscore the importance of integrating visualization techniques in geometry instruction, particularly for MLs, who may encounter additional linguistic barriers. Visualization appears to be the key to bridging the gaps in comprehension by providing a shared, non-verbal medium for interpreting and engaging with geometry concepts during problem-solving. However, geometry vocabulary was found significant during problem-solving as it affected ML understanding and visual thinking during problem-solving. The findings underline that while visualization aids comprehension for MLs, it does not function independently of language. Geometry instruction in multilingual classrooms must consistently incorporate annotated diagrams, interactive drawing tasks, and vocabulary-focused prompts to bridge linguistic gaps and support deeper conceptual understanding. This aligns with the Commognitive perspectives, which emphasize the importance of word use and visual mediators in facilitating mathematical thinking and communication.

The research contributes to understanding how cognitive and linguistic factors intersect in multilingual classrooms and suggests pedagogical strategies that leverage visualization to support learning. One limitation of this study is the small sample size and the single school context, which may affect the generalizability of the findings. Moving forward, future research could explore the efficacy of visual-based interventions to support MLs or focus on implementing teacher professional development programs that specifically address the needs of such classrooms, potentially improving teaching practices and learning experiences in diverse linguistic environments.

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