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Early ways of relating quantities from a sixth grader

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Abstract

This article reports on a research project that focused on exploring the early forms of relating quantities in situations of variation exhibited by a sixth grade student. For this exploration, a case study approach was used, with a task-based Interview applied to collect data, and thematic analysis utilized to process the data. Relating quantities refers to characterizing and representing the changes of variables that vary among themselves. The early forms of relating quantities that emerged include variable identification, general recursion, correspondence, pre-coordination, and gross coordination of values. The correspondence relationship was the most frequently observed, while pre-coordination and gross coordination were the least common, suggesting that the latter could be studied in greater depth among young students.

Keywords: relating quantities, correspondence, sixth grade, covariation

INTRODUCTION

This work focuses on the early forms of relating quantities or elements of a system in situations of variation. It is common for individuals to encounter phenomena of change and variation that occur in the social or natural contexts in which they operate. To understand and explain these phenomena, the development of perceptual and reasoning skills is essential. These processes can be nurtured within school contexts, where mathematics and science provide the curricular space to develop this type of knowledge. In the field of mathematics education, this is often referred to as variational thinking, functional thinking, or covariational reasoning. Cantoral (2019) reflects on the forms of mathematization and their support in the social field regarding the variational practices that should be considered in classrooms and laboratories where scientific knowledge is fostered.

To address change and variation, we employ forms of functional thinking aimed at constructing, describing, and reasoning about relationships between quantities that change together. We strive to engage students in processes that generalize relationships between quantities, encompassing their representation and the ability to reason about them (Stephens et al., 2017) in order to interpret and predict the behavior of phenomena characterized by change and variation. An example of a context from which ways of relating quantities can emerge is when we read in the news: “the sea level is rising as the oceans continue to absorb heat from the atmosphere” (Johnson, 2023, p. 17). To understand this statement, it is crucial to clarify how the rise in sea level, the increase in temperature, and the elapsed time are related as references for comparison in the phenomenon of change being discussed.

In the realm of school mathematics, interpreting relationships between quantities or explaining how elements interact represents a potential foundation for developing reasoning such as quantitative and covariational reasoning (Johnson, 2023). According to Thompson and Carlson (2017), such reasoning positively impacts the understanding of mathematical concepts such as rate of change, ratio of change, and functions. This is why research on functional relationships (e.g., Blanton & Kaput, 2004; Johnson, 2023) has examined the advantages that students can gain from developing these forms of reasoning.

Research investigating functional thinking in primary school children suggests that they should be given opportunities to work on interpreting the relationship between quantities that change together from an early age (Blanton et al., 2015; Stephens et al., 2017; Tanışlı, 2011). Furthermore, it has been recommended that mathematics programs for this educational level should include functional thinking,

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Contribution to the literature

- This article exemplifies the functional relationships reported in other studies conducted in the field of early algebra with elementary school students.
- It provides elements that help us understand how different ways of relating quantities are interconnected and addresses some aspects that are often not covered in specialized literature.
- The article highlights that our student demonstrated only limited instances of covariational relationships and analyzes the factors that could promote their development.

which aims to address the relationship between variable quantities (Blanton & Kaput, 2004).

Attention to the simultaneity of changes among quantities has also been referred to as covariational relationships. Research investigating functional thinking in young learners (e.g., Blanton et al., 2015; Panorkou & Maloney, 2016; Stephens et al., 2017) has identified covariational thinking as an early way of relating quantities. Covariation, as a form of reasoning, has been defined more broadly in research with pre-college and college students (e.g., Carlson et al., 2002). Covariational reasoning has been characterized as a cognitive skill that may enhance students' and teachers' mathematical understanding (Thompson & Carlson, 2017). Moreover, covariational reasoning is argued to represent a complex cognitive skill, even for outstanding students in calculus courses (Carlson et al., 2002).

In a literature review described in the following section, it was found that there are few investigations that have studied the relationship between quantities and covariation in young students over the last ten years (Blanton et al., 2015; Panorkou & Maloney, 2016; Pittalis et al., 2020; Stephens et al., 2017). This indicates that there are areas in this field that have not been sufficiently explored. For example, little is known about how to lay the groundwork in elementary grades to develop more sophisticated reasoning, such as covariation, when students reach middle or higher levels. These limitations in research may stem from the belief that studying functional relationships that encompass the joint variation of quantities requires a level of formal abstract thinking that is only accessible from the secondary level onward (Blanton et al., 2015). Although it has been demonstrated that preschool and primary school students are capable of establishing covariation relationships between quantities and representing these relationships (Panorkou & Maloney, 2016; Pittalis et al., 2020; Stephens et al., 2017), this study raises the research question: what early forms of relating quantities does a sixth-grade student develop when working with covariant quantities?

LITERATURE REVIEW

Research in mathematics education that investigates quantities that change together has primarily been grounded in covariational reasoning theory (Carlson et al., 2002) and has focused on secondary-level students aged 13-15 (e.g., Jacobson, 2014; Johnson et al., 2017; Wilkie, 2019), pre-university students aged 16-18 (e.g., Ferrari-Escolá et al., 2016; Johnson, 2015; Şen Zeytun et al., 2010), and university students (e.g., Carlson et al., 2002; Kertil et al., 2019; Paoletti & Moore, 2017).

The emphasis on covariational reasoning beginning in seventh grade may be attributed to the school curriculum's introduction of formal function concepts at this stage. This reasoning has been recognized as foundational for understanding these concepts (Thompson & Carlson, 2017). In this context, Confrey and Smith (1994, 1995) argue that the covariational approach to teaching functions is more productive than the traditional correspondence approach, which has dominated conventional function curricula.

Thompson and Carlson (2017) provide a historical overview of the study of functions and highlight covariational reasoning as a theoretical construct that underpins this mathematical concept. They assert that covariational reasoning is essential for the mathematical development of both students and teachers, as it allows access to productive conceptions in the study of functions. Research involving young students (ages 3 to 12) has characterized the covariation relationship as an early way of relating quantities.

Studies focusing on preschool and elementary students' conceptualization of relationships among quantities that change together utilize various terms to describe the coordination of simultaneous changes among quantities or elements of a system. Panorkou and Maloney (2016) refer to this mental activity as the functional covariation relationship, building on the work of Confrey and Smith (1995) to describe how changes in one quantity in a numerical pattern relate to changes in another quantity in a different pattern.

Stephens et al. (2017) and Pittalis et al. (2020) refer to these cognitive processes as covariation thinking. In their work, Stephens et al. (2017) characterized the progression of students from third to fifth grade in the generalization and representation of functional relationships, proposing levels of sophistication to describe students' algebraic thinking. They identify covariation thinking as level 3, which involves recognizing the coordinated relationship between two variables rather than treating them separately.

Pittalis et al. (2020) propose a theoretical model describing the functional thinking modes of first to third-grade students and identifying covariational thinking as one of these modes. Drawing on the work of Smith (2017) and Thompson and Carlson (2017), they define covariational thinking as the ability to analyze how the values of two quantities vary simultaneously, leading to the establishment of a rule that governs their relationship.

Definitions used in research with young children regarding functional covariation relations (Panorkou & Maloney, 2016) or covariation thinking (Pittalis et al., 2020; Stephens et al., 2017) align closely with research on covariational reasoning conducted at higher educational levels. In both contexts, coordinating simultaneous changes between quantities is emphasized as a central action (Carlson et al., 2002). Notably, while research in elementary grades has not centered on covariational reasoning as a primary objective, findings have indicated that young students develop covariation relationships as they conceptualize the joint growth and variation of quantities (Kaput, 1994, as cited in Ellis et al., 2023).

The literature review indicates that covariational thinking among young students represents an emerging line of research, one from which explanations could be generated regarding how these relationships arise and how their development may impact mathematical learning at both elementary and higher educational levels.

THEORETICAL REFERENCES

Early Ways of Relating Quantities

Early forms of relating covariant quantities refer to the ability of preschool and primary school students to reason, describe, and represent the behavior of two or more variable quantities (Blanton & Kaput, 2004; Stephens et al., 2017). This notion aligns with Johnson's (2023) assertion of an intellectual need to establish relationships, as young students' initial experiences often focus on explaining how these elements work together as a system before engaging in formal experiences with established quantities like rates or ratios of change. Thus, these early forms of relating quantities can support the development of young children's quantitative and covariational reasoning (Johnson, 2023).

The processes of reasoning, describing, and representing the behavior of quantities or elements that interact are triggered by the question: What happens to one quantity when the other changes? Understanding how multiple quantities that change in relation to each other behave involves representing these relationships using natural language, formal algebra, tables, or graphs (Stephens et al., 2017). For example, measuring the performance of a moving vehicle necessitates considering both the distance it travels and the amount of fuel consumed at all times during its motion.

Covariation Relation

In this work, the covariation relationship is understood in the same manner as described by Confrey and Smith (1995), who outline two types of functional relationships. The first, the correspondence relationship, focuses on relating two data sets through a rule that allows for finding values of y or $f(x)$ given a specific x, as in the equation $f(x) = 2x + 1$. The second type refers to the covariation relationship, which is characterized as a relationship between quantities in two data sequences, where the changes in one quantity occur simultaneously with the changes in another.

A functional covariation relationship is established when individuals conceptualize how the values of two quantities vary together, analyzing and identifying the nature of these simultaneous changes and incorporating that understanding into their description of the functional relationship (Confrey & Smith, 1995). When working with tables containing corresponding pairs of covariant quantities, this relationship involves treating each column as a set of variable quantities, identifying variations within each column, observing patterns in the table, and coordinating the variations in both columns by moving vertically up or down in the table (Pittalis et al., 2020; Tanışlı, 2011).

In this study, functional covariation thinking is viewed as an early way of relating quantities, defined as the ability to identify and represent the values of two covariant quantities or elements, either in numerical or figurative sequences, as well as in value tables. With this approach, students can establish function rules to explain relationships, such as: when one quantity increases by 1, the other quantity increases by 2, and they can also use this rule to calculate successive covariant elements or quantities in data sequences.

MATERIALS AND METHODS

To investigate early forms of relating variable quantities, a case study approach was utilized. Data were collected through a task-based interview, followed by analysis using thematic analysis techniques.

The Case Study

Case studies have gained acceptance as a research method in the scientific community (Yin, 2006) because they facilitate the understanding of participants' thought processes by employing various sources such as documents, direct interviews, and participant observations. These methods are focused on the phenomenon under study and its contextual development. Implementing qualitative research, the case study can be exploratory, aiming to bridge theories within the theoretical framework and the reality being examined (López González, 2013). Single case studies are characterized by concentrating the analysis on a single case, with their validity rooted in the critical nature that allows for confirming, altering, or expanding knowledge about the subject matter (López González, 2013).

This paper studies the case of Lizeth (pseudonym), a high-performing sixth-grade student among a group of 20 children. Lizeth was selected due to her demonstrated ability in solving tasks involving the joint variation of quantities or elements of a system. She expressed a willingness to participate in the research, and her parents provided consent.

Context and Participants

The research was conducted in an elementary school on the outskirts of Chilpancingo, Mexico. The student body comprises children from various social strata, with many coming from impoverished families seeking better opportunities in the city, alongside a significant number from middle-class families employed in bureaucratic and service sectors. The curriculum followed by the children corresponds to the new study plan implemented by the Government of Mexico in 2022, known as the New Mexican School (SEP, 2022), which aims to connect educational content with the realities of the students' communities.

This curriculum is structured around four formative fields: languages; knowledge and scientific thought; ethics, nature and societies; and human and community. The mathematics content falls under the knowledge and scientific thought field. One of its objectives is to enhance students' understanding of natural processes and phenomena in relation to social contexts through interpretation, systematization, representation with models, and argumentation (SEP, 2022). For fifth and sixth grades, the curriculum emphasizes the importance of mathematics in knowledge construction, encouraging students to articulate their learning verbally and in writing using both words and symbols.

For the specific purposes of this research, the curriculum provides opportunities for children to interpret phenomena in their environment, including phenomena of variation and change, which they are encouraged to interpret both quantitatively and qualitatively. This includes recognizing patterns of change, establishing rules to calculate missing values, and representing and explaining how quantities that change together are related.

Task-Based Interview Design

Task-based interviewing was employed to collect data. This method is a specialized form of clinical interview that allows participants to interact not only with the interviewer but also with a carefully designed task environment (Goldin, 2000). This approach enabled direct interaction with the subject of study while working on mathematical tasks in a pencil-and-paper context, as suggested by Maher and Sigley (2020). This format allowed Lizeth to represent her ideas and mathematical reasoning regarding how she relates quantities or elements of a set. The interview comprised mathematical tasks and semi-structured questions to facilitate interaction between the interviewer and the student, serving as a mediation mechanism (Goldin, 2000).

The tasks were designed based on several criteria: the cognitive level of the participating student, insights from the specialized literature on how primary school students conceptualize and represent functional relationships, and the ability of the tasks to encourage thinking about the joint variation of quantities or elements.

Five tasks of two types were created and validated: figural sequences and contextual situations. In task 1, a sequence of six figures is presented, with observable patterns of change between figures. The participant was asked to identify the changing elements, the pattern of change, and to predict subsequent figures in the sequence based on their relationships and positions. Task 2 and task 3 involved geometric elements (squares and cubes) and required students to identify behavior patterns and relate the number of elements to their positions in the sequence. These tasks also encouraged students to formulate general rules to find elements corresponding to close and distant figures in the sequences.

Task 4 and task 5 presented contextual problems designed to help participants establish relationships between quantities that change together. Task 4 involved the relationship between the number of people and time, while task 5 addressed kilometers traveled and liters of fuel consumed. Participants were tasked with organizing data into numerical tables and explaining the relationships between the data in both columns.

The validation process for the interview instrument included designing ten initial tasks applied to five fifth and sixth-grade students in separate sessions. This approach assessed the relevance of the tasks, allowed for adjustments, and led to the selection of the most productive tasks from the research team's perspective. The tasks were designed based on four criteria: accessibility for sixth-grade students' cognitive development; similarity to tasks typically presented in current textbooks; potential to encourage the use of early ways of relating varying quantities; and alignment with tasks utilized in related research.

Data Analysis

The interview with the student was video recorded, and the resulting material was transcribed, including her arguments and questions posed by the research team for

Table 1. Data analysis process*

Task	Extract from the interview	Response pattern	Topics (form of relationship)	Code
T1	S: In the star section, one is illuminated and one is not, in the arrow section, the position of the one pointing changes. R: What is changing in this sequence? S: The position of the arrows and how the stars are illuminated.	Identify what changes	Identification of variables	IV
T2	S: The number of the position must be multiplied by itself, for example: position 10, multiply 10 by 10, equals 100. R: What figure would occupy the 20th place? S: It would be 400 squares. R: Why? S: Multiply 20 by 20 to get 400.	Calculate missing values	Correspondence	C

Note. *The phrases in italics are parts of the answers associated with a form of relationship; T1: Task 1; T2: Task 2; R: Researcher; & S: Student

Table 2. Ways of relating quantities (n = 5)

Form of relationship	Core actions	Code	Task	Frequency	Total
Identification of variables	Identify variables involved in a variation situation.	IV	1, 2, 3, 4, 5	1, 1, 1, 1, 2	6
General recursion	Identify how changes behave (recursive pattern)	GR	1, 2, 3, 5	3, 1, 1, 1	6
Correspondence	Performs operations to calculate missing values.	C	1, 2, 3, 4, 5	2, 3, 6, 2, 5	18
Pre-coordination	Identifies the variation of two variables asynchronously.	PC	2, 3	1, 1	2
Gross coordination	Understand how the values of quantities vary from each other.	GC	4	2	2

further investigation. The written responses provided evidence of the problem-solving procedures, which were digitized and incorporated into the transcribed interview texts. For data analysis, the thematic analysis approach suggested by Braun and Clarke (2012) was employed to identify patterns in the ways of relating quantities (themes). This involved using the data from task responses and arguments expressed during the interview. Several themes were identified, some of which are exemplified in **Table 1** and will be referred to as ways of relating quantities in the results report. The analysis followed Braun and Clarke's (2012) processes of familiarization, coding, and identification.

Thematic analysis was conducted in five phases:

(1) familiarization with the data through repeated readings of the transcripts,

(2) establishing initial codes from participants' verbal expressions or operational actions in their task responses,

(3) searching for themes by comparing and compiling the initial codes into overarching themes defined as early ways of relating quantities,

(4) reviewing the themes by matching data from task responses with interview arguments, and

(5) defining and naming the themes, which represent the ways of relating quantities discussed in the results report.

To enhance objectivity in the results during the first four phases of thematic analysis, researchers independently worked with the data before comparing and discussing their findings as part of a triangulation process (Flick, 2004) to mitigate potential bias from a single researcher. In cases of disagreement, joint sessions were held to analyze the data and reach consensus. **Table 1** presents examples of theme detection that guided the identification of early ways of relating quantities.

RESULTS

Five early forms of relating variable quantities were found: variable identification, general recursion, correspondence, pre-coordination and gross coordination. **Table 2** shows the forms of relating quantities or elements of a set that were identified, as well as the central actions that support these forms of relationship, the codes with which they are identified, the tasks in which they appeared and the frequency with which they appeared.

Identification of Variables

The identification of variables appeared in the responses to the five tasks. In tasks 1, 2, and 3, where Lizeth was asked to observe, explain and draw the figures that follow in the sequences, in her responses she was able to identify the variables involved in the sequences, for example, in her response to task 1 she wrote what is shown in **Figure 1**.

During the interview, Lizeth's responses indicate that she is able to identify the variables, as shown in the following extract:

> Interviewer (R): Can you explain to me what is changing in this sequence?

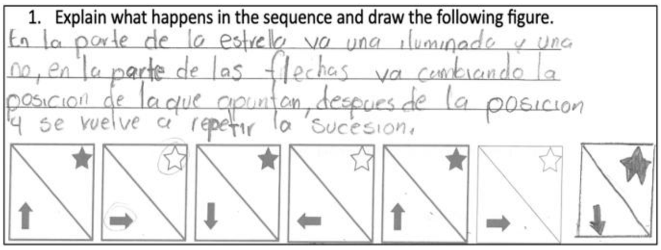


Figure 1. Identification of variables in the succession (Source: Authors' own elaboration)

Figure 2. Location of variables in a record table (Source: Authors' own elaboration)

> Lizeth (S): The position of the arrows and how the stars are illuminated.

For task 2, when asked to explain what she observes in the sequence of squares, Lizeth responded:

> R: Explain your answer.
>
> S: In this series of boxes that you are adding to know the number of boxes that correspond to it, you must multiply the number of the position by itself.

As can be seen in her answers, Lizeth identifies that the variables present in the sequences are the number of elements in each figure and the position of the figures in the sequence.

In task 4, Lizeth was asked to create a table to record the data for the first question. She wrote what appears in **Figure 2**. She called the first column "number of people" and the second column "time." For her, the variables that are changing are the number of people and the time spent by the people cleaning the room. These productions indicate that Lizeth managed to identify the variables involved in the proposed task.

General Recursion

General recursion emerged in Lizeth's responses to tasks 1, 2, 3, and 5. In task 1, this relationship was evident when Lizeth identified the recursive pattern in the sequence of stars and arrows. She stated, "... after position 4, the sequence is repeated" (see **Figure 1**). Her recognition of this recursive pattern enabled her to predict the next figure in the sequence. The following exchange illustrates this process:

> R: Regarding those changes that you observe in the direction of the arrows and the color of the stars, can you predict which figure would follow?

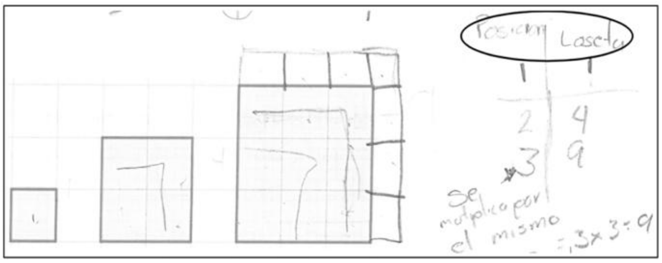


Figure 3. Response to task 2 (Source: Authors' own elaboration)

> S: Yes.
>
> R: Explain to me, which would follow?
>
> S: An illuminated star and an arrow pointing downwards would follow.
>
> R: What makes you think that figure would follow?
>
> S: Because from the 4th figure onwards, the succession of arrows is repeated.

When predicting the next figure in the sequence from task 1, Lizeth engaged in a generalization process. She observed that the star in the first figure is colored, while the next one is white, and recognized that this pattern would continue throughout the remaining figures in the sequence. Similarly, she noted the change in the direction of the arrows and that this pattern repeats after the fourth figure. Through these identification actions, Lizeth successfully established the recursive pattern.

In task 2, when the interviewer asked Lizeth to explain her observations in the sequence of squares (see **Figure 3**), she responded:

> S: 1 square, 4 squares, 9 squares, and the next would be … four ... four (counting the number of squares in a row and a column) 16.

By articulating her observations in the sequence, Lizeth attempted to identify the behavior of the consecutive elements. This approach demonstrated her commitment to explaining how changes occur, which assisted her in determining unknown terms in the sequence.

Lizeth employed counting as a strategy to arrive at general recursion, which was evident in her responses to task 3 (see **Figure 4**). For each element in the sequence, she made numerical records to keep track of the number of cubes contained in each figure. Once she identified how the number of cubes increased from one term to another, Lizeth was able to ascertain the number of cubes that other figures in the sequence would contain.

As shown in Lizeth's answers, this way of relating quantities that we have called general recursion allowed

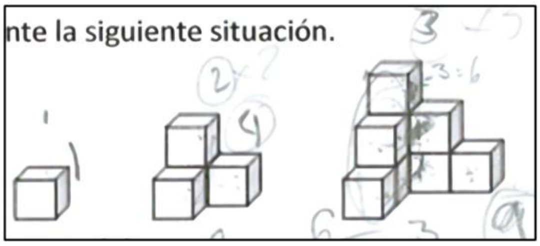


Figure 4. Counting as the basis of general recursion (Source: Authors' own elaboration)

her to follow a process of evolution to establish other ways of relating quantities or elements of a system.

Correspondence

Correspondence as a way of relating quantities or elements of a sequence appeared in Lizeth's answers across all five tasks posed in the interview. In task 1, when asked what figures 16 and 18 in the sequence would look like, Lizeth responded as follows:

> R: What would figure 16 look like?
>
> S: Like 4.
>
> R: Would figure 16 be like 4? Why?
>
> S: Because 4 times 4 is 16.
>
> R: And figure 18, what would it look like?
>
> S: It could be like this one (points to figure 2).
>
> R: It would be like 2, but what does that make you think?
>
> S: … Yes, because 4 and 4 (referring to repeating the first four figures 4 times or multiplying 4 by 4) are 16 and 2 is 18.

To establish a correspondence relationship, Lizeth relied on numerical calculations, as seen when she multiplied 4 by 4. She focused on an additive operation of repeating the first four figures in the sequence, concluding that figure 4 would also occupy position 16. This same reasoning enabled her to deduce that position 18 corresponds to a figure similar to that of figure 2, again using multiplication.

Correspondence as a form of functional relationship is expressed through an algebraic rule (Confrey & Smith, 1995). This rule allows for calculating unknown values in data sets, where a value in one set corresponds to a unique value in another (Panorkou & Maloney, 2016). In task 1, Lizeth related the data sets of the sequence, employing correspondence to determine what a figure in the sequence would be for a given position. When asked to identify figures at positions 16 and 18, her reasoning involved multiplying by 4 (the number of varied elements) to reach the requested position. It should say: In the case of figure 16, the student solved x= 4 X 4; in the case of figure 18, the student solved x= (4 X 4) + 2.

In task 2, Lizeth established a correspondence relationship by constructing a two-column table (see **Figure 4**) representing the position in the sequence and the number of squares corresponding to each figure. She referred to the squares of each figure as "tiles" (floor tiles).

In this response, Lizeth proposed a general rule or formula for determining the number of squares for different figures in this sequence, as illustrated in the following exchange:

> S: In this series of boxes that you are adding to know the number of boxes that correspond to it, you must multiply the position number by itself. For example: position 10, you multiply 10 by 10, which equals 100.
>
> R: And what figure is next?
>
> S: From this, figure 4 would follow.
>
> R: How many boxes would figure 4 have?
>
> S: 16.
>
> R: What figure would occupy place 20?
>
> S: It would be 400 boxes.
>
> R: Why?
>
> S: Because you multiply 20 by 20 to get 400.

The rule Lizeth established and expressed first verbally and later in numerical form is of the type ($f\,[x] = x^2$), demonstrating her understanding that to find the number of squares corresponding to a given position, one must square the position number. Additionally, Lizeth was able to articulate how many squares the rows and columns of each figure would contain, evident when she discussed the number of squares in figure 20.

It is noteworthy that the correspondence relationship Lizeth established in task 2 served as a mechanism that extended to construct arguments and structure her response to task 3, given that both tasks share a similar nature. As seen in the following interview excerpt, the correspondence rule established in task 2 remained applicable and reinforced her approach to relating the elements of the sequences presented in task 3:

> S: This is a sequence of cubes that multiplies the position by itself. For example: position 5, 5 times 5 equals 25.
>
> R: And in this case? (points to figure 3).

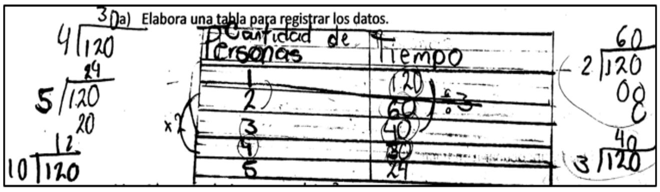


Figure 5. Correspondence relationship in tables of values (Source: Authors' own elaboration)

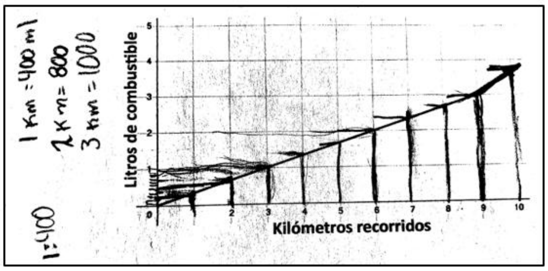


Figure 6. Plot of corresponding values in the graph (Source: Authors' own elaboration)

> S: In position 3, there are 2 here, 3, 5, 6, 7, 8, 9, there are 9. So, 3 times 3 equals 9.

The correspondence relationship Lizeth established in task 2 and task 3 was integral to her reasoning process, enabling her to determine the number of elements corresponding to figures based on their positions in the sequence.

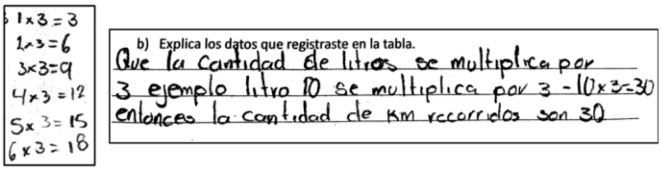
For task 4, Lizeth created a table of values and using the data from the first column determined the corresponding values for the second column by employing numerical calculations to divide 120 minutes among different values for the number of people (see **Figure 5**).

In task 5, when Lizeth was asked to explain her answers, she relied on the graph to match the kilometer values with the fuel liter values (see **Figure 6**), so she drew lines with her pencil to mark the vertical displacements (kilometers) and their corresponding horizontal displacements (liters). **Figure 6** shows the corresponding values in the graph.

During the interview, Lizeth explains that for values less than 3 kilometers, values less than 1 liter of fuel correspond. In addition, she explains that for values 3, 6 and 9 kilometers, 1, 2, and 3 liters correspond, respectively, as shown in the interview extract:

> S: For example, here it is less than 1 liter of fuel (points to notes made on the task chart), and at 3 it starts with 1 liter of fuel; at 6 the same; at 9 another.

> S: For every 3 kilometers travelled it is 1 liter of fuel. Here it is 1 liter, 6 it is another liter, and so on.



a b

Figure 7. Rule for calculating corresponding values (Source: Authors' own elaboration)

When Lizeth uses numerical correspondence to relate the values of the two variables, in this case, kilometers and liters, where each amount of liters consumed by the truck corresponds to a number of kilometers traveled. By engaging in this form of relationship, Lizeth was able to establish the general rule to calculate the values for kilometers by multiplying the amount of liters by 3 (see part a and part b in **Figure 7**), that is, the student is thinking of a formula of the type $f(x) = 3x$.

Pre-Coordination of Values

In task 2 and task 3, sequences of figures were presented (see **Figure 3** and **Figure 4**), and Lizeth was asked to explain whether there was any regularity or pattern. Her responses provide insights into her way of relating the variables through pre-coordination of values. For example, when asked about the pattern or regularity in the sequence, she stated: "when the position changes, the number of squares also changes" (task 2) and "when the position changes, the number of cubes changes" (task 3).

Lizeth's responses in these statements from task 2 and task 3 indicate that she conceives of the variables changing asynchronously. That is, Lizeth understands that the position of the figures in the sequence changes first, and then the number of elements in each figure changes.

Gross Coordination of Values

In Lizeth's responses for task 4, evidence was found that she managed to relate the quantities through gross coordination of values. When asked to explain her solution process, she stated:

> "I found that, for example, I don't know, 2 is multiplied by 2 ... no, it is added, 2 plus 2 by itself gives 4. So, it is half, for example: if the time for 2 people is 60 minutes, that of 4 is 30."

In this response, Lizeth sought to explain how the two variables are related. She understood that as the number of people increases, the time decreases: "it is multiplied, it is added" and "it is half."

This same way of relating the variables appeared again in Lizeth's written response to part (b) of the task, which asked "How are these data related?" (see **Figure 8**).

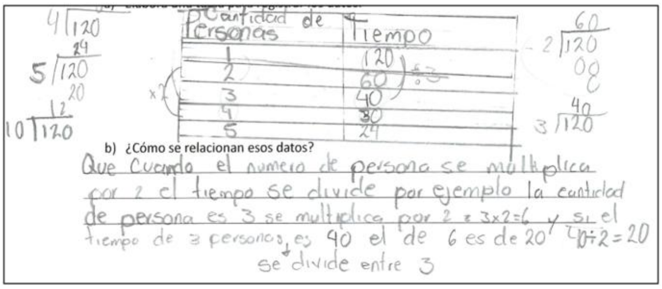


Figure 8. Arguments supporting the gross coordination of values (Source: Authors' own elaboration)

In this answer, Lizeth reaffirms that as the number of people increases (first column), the time decreases (second column). The ideas expressed by Lizeth indicate that she was able to conceive of the inverse relationship between these variables - as the number of people increases, the amount of time decreases. In her example, she states that as the number of people increases from 3 to 6, the time decreases from 40 to 20 minutes. This demonstrates that Lizeth understands how the two variables are related and that they vary inversely with respect to one another.

DISCUSSION

The results indicate the use of five ways of relating quantities: variable identification, general recursion, correspondence, pre-coordination of values, and gross coordination of values.

Lizeth's identification of variables was possible in all the proposed tasks. This is a fundamental skill in which it is possible to develop reasoning related to functional thinking. While the nature of variables in tasks has sometimes been taken for granted (Leinhardt et al., 1990), this work suggests that it can be stimulated by presenting students with iconographic sequences (sequences of figures) that give them the opportunity to focus on the attributes of the figures, as suggested by Johnson (2023). In Lizeth's case, the identification of variables in task 1 was possible when she compared the attributes of the figures: the coloring of the stars and the orientation of the arrows.

Task 2 and task 3, consisting of sequences of squares and cubes, lent themselves to quantifying the elements and forming numerical sequences: 1, 4, 9, ...; 1, 4, 9, 16, ..., which could then be associated with the order or place they occupy in the sequence. Lizeth used a table of values in task 3 and labeled the figures in task 4. Here, the idea of variables associated with numbers (order and quantity) was evident, and Lizeth was able to analyze the sequence of figures and compare the antecedent elements with the subsequent ones. When Lizeth identified the variables in the situations, her attention was directed to establishing a relationship between these variables, which is associated with the idea of function as correspondence, considered the basis of functional thinking (Smith, 2017). In these tasks, the identification of the variables was necessarily linked to the correspondence relationship between them.

Unlike the others, task 4 presented a contextual situation in written form and asked Lizeth to make a table and analyze how the data are related. When building the table, Lizeth had no difficulty using the variable "number of people" and assigning a progressive natural number (1, 2, 3, ..., 5). However, when determining the corresponding value in the other column (the time to complete the task), she found it necessary to resort to a pattern (dividing 120 minutes by the number of people) because the task required it. Task 5 presented a Cartesian graph, and the evidence indicates that the extraction of the variables did not require much effort for Lizeth; she was able to identify and transfer them when organizing the table, placing the quantity in liters in one column and the corresponding number of kilometers in the other column.

Three main characteristics are noted in the productions for task 4 and task 5. First, the use of sequences to denote changing elements or quantities, such as the number of people and time in task 4, and the number of liters and kilometers in task 5. Second, the need to relate the elements or quantities through correspondence. And third, the need to resort to a pattern to determine the values of the dependent variable. These results are similar to those reported by some researchers (e.g., Blanton et al., 2015; Brizuela et al., 2015; Moss & McNab, 2011; Pittalis et al., 2020; Sfard, 2012), suggesting that students in the last grades of primary school (3rd to 5th grade) can understand and take advantage of variables in the generalization process, as evidenced by Lizeth's discovery and use of patterns to determine the value of the dependent variable.

General recursion appeared in tasks 1, 2, 3, and 5. When Lizeth discovered the recursive pattern, her attention focused on identifying "what is repeated," which enabled her to draw missing figures in the following positions of the sequences, as in the case of task 3 where she counted the elements and kept records to understand the behavior of the known terms. In task 5, her attention was directed to the behavior of the graph, and she identified that each time the fuel increased by three liters, this pattern would continue for the rest of the points. These results are similar to those reported in other research (e.g., Blanton et al., 2015; Pittalis et al., 2020; Stephens et al., 2017), which found that elementary school students exhibited recursive thinking when using contextual and pre-symbolic strategies that led them to establish generalization processes.

Pittalis et al. (2020) suggest that recursion could help develop correspondence and covariational thinking, as students become aware of the structure in the patterns and generalize about it. The evidence in this research is consistent with this statement regarding the correspondence relationship. In the results of tasks 1, 2, and 3, where sequences were presented, Lizeth identified the number of elements of the figures and their respective positions. This was not the case with the covariation relationships that appeared in task 4 and task 5, as Lizeth did not focus on identifying the behavior of figural elements in relation to their position in the sequences. As other research has suggested (e.g., Blanton et al., 2015; Blanton & Kaput, 2004), the early development of covariation may depend on the opportunities provided to students to interact with contextual tasks and verbal statements, which can help them transcend figural and numerical sequences and favor the identification of behavioral patterns and, consequently, the general recursion relationship.

Regarding correspondence, it appeared in all five tasks. In task 1, when Lizeth was asked about the appearance of some figures in the sequence, she performed numerical calculations (multiplication and addition) to construct her answers. In task 2 and task 3, her main action was to construct a rule that allowed her to correspond values, expressing that to know the number of elements in each figure, she had to multiply the position number by itself. For task 4 and task 5, she presented her rules for calculating and corresponding values (multiplying or dividing) in numerical form, thus relating to the variables involved and recording values in the second column of the tables she constructed. Similar to the study by Pinto et al. (2016), our results provide evidence that Lizeth used correspondence more frequently, although unlike the cited study, this relationship was not limited to working with figural sequences but was also extended to task 4 and task 5, which involved contextual approaches.

When Lizeth expressed rules to calculate values of the variables, her answers seem to align with Confrey and Smith's (1994) description of the correspondence relationship. According to these authors, the function correspondence approach is characterized by constructing a rule to relate the values of one variable with the values of the other variable, similar to what Lizeth expressed when working with quadratic growth sequences (task 2 and task 3). This contrasts with the findings of Stephens et al. (2017), who reported that third to fifth-grade students expressed correspondence rules symbolically first (using variables) and in words later. In Lizeth's case, she first wrote her correspondence rule in words in her answers to task 2 and task 3, perhaps influenced by the task statements that explicitly required it.

Regarding the covariation relationships, pre-coordination of values and gross coordination of values (Thompson & Carlson, 2017), they were less frequently observed. Pre-coordination of values was present in task 2 and task 3, where Lizeth expressed that when the position in the sequence changes, the number of elements in the figures also changes. In the case of gross coordination of values, it was observed in task 4, where Lizeth seemed to understand that as the number of people increases, the amount of time decreases.

Unlike the findings of Panorkou and Maloney (2016), who reported that fifth-grade students developed skills to recognize and establish pairwise covariation relationships in numerical patterns, these forms of covariation relationships occurred less frequently in Lizeth's case. The difficulties Lizeth showed in covariational reasoning may be explained by the proposal of Blanton et al. (2015) that this type of functional relationship represents more sophisticated forms of thinking. Additionally, as Pittalis et al. (2020) suggest, the development of the covariation relationship can be a multifaceted process. Before reaching gross coordination of values, Lizeth was engaged in the identification of variables, general recursion, and even establishing correspondence between the variables when organizing her value table in task 4.

Pre-coordination of values and gross coordination of values appear as the most elementary levels in the covariational reasoning framework suggested by Thompson and Carlson (2017). In this research, these levels of reasoning helped characterize two early ways of relating quantities. Based on the observations of Lizeth's responses in task 4, it seems that gross coordination of values can be favored through contextual problem-solving tasks. Although task 4 was posed in the discrete domain and may have limitations for transitioning from coarse images of change to smooth images of change (Castillo-Garsow, 2012), we consider that for elementary students, contextual tasks (Panorkou & Maloney, 2016) may be more relatable to their classroom experiences and can be useful for conceptualizing functional thinking in young students (Pittalis et al., 2020).

CONCLUSIONS

As reported in the literature, tasks that ask students to deduce patterns are traditionally used at the preschool and primary school levels, mainly contributing to the development of general recursion as a way of relating quantities (Pittalis et al., 2020). Although recursive patterns favor generalization processes and could help develop correspondence and covariation relationships, we consider the need to work with tasks specifically designed to promote covariation relationships and support students in conceptualizing jointly varying variables, which, according to Johnson (2023), represents a fundamental mathematical idea that should not be reduced to formal definitions.

The objective of reporting the five ways of relating to quantities exhibited by Lizeth has both practical and theoretical implications. Practically, it provides insights into the characteristics of tasks that helped Lizeth develop each form of relationship. Theoretically, it contributes to characterizing the five early ways of relating quantities, which can serve as a basis for further investigations on the functional thinking of elementary school students. Based on the experience gained in this research, we join Johnson's (2023) call that before addressing the formal study of different types of functions, it is necessary to focus the work with students on the relationships between attributes or quantities that change simultaneously.

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