MODESTUM
EURASIA Journal of Mathematics, Science and Technology Education, 2025, 21(6), em2656
ISSN:1305-8223 (online)
OPEN ACCESS
Research Paper
https://doi.org/10.29333/ejmste/16540

Design and implementation of a task sequence for teaching homothety

Yenifer Aguilera Moraga [1], Paula Verdugo-Hernández [2*], Jessica Gatica [3]

[1] Pedagogía en Educación Media en Matemática y Física, Universidad de Talca, Linares, CHILE
[2] Escuela de Pedagogía en Ciencias Naturales y Exactas, Facultad de Ciencias de la Educación, Universidad de Talca, Linares, CHILE
[3] Instituto de Matemáticas, Universidad de Talca, Linares, CHILE

Received 03 March 2025 ▪ Accepted 19 May 2025

Abstract

The teaching of geometry in Chile faces several challenges, as evidenced by the low performance of students in international assessments. In particular, the concept of homothety is impacted by teaching methodologies that emphasize rote memorization and procedural repetition rather than conceptual understanding. This study explores how a task sequence fosters mathematical work with the concept of homothety among 15- to 16-year-old students in a public high school in the Maule Region of Chile. Grounded in the mathematical working space (MWS) theory, this research provides a framework for analyzing how students engage with mathematical tasks. Employing a qualitative approach, specifically a case study, the findings reveal that the task sequence contributes to progressive mathematical reasoning. The study concludes that the designed activities align with the criteria for "emblematic tasks", evidencing their fundamental role in supporting students' conceptual understanding of homothety.

Keywords: teaching, task design, homothety, mathematical work, personal MWS

INTRODUCTION

International assessments, such as the 2022 program for international student assessment, underscore the persistent challenges faced by Chilean students in mathematics, with performance levels declining compared to previous years. The data reveal that 44% of students performed at a deficient level, evidencing only the ability to represent simple mathematical situations without explicit guidance. In contrast, only 1% reached the highest proficiency level, which entails modeling complex situations and identifying appropriate solution strategies (OECD, 2023). Similarly, the 2019 trends in international mathematics and science study reported comparable findings, positioning Chile 59 points below the international average. Notably, only 1% of students attained an advanced level, with the study further highlighting that performance in geometry was significantly lower than the overall mathematics score (Agency for Quality Education [Chile], 2020).

These results may be linked to dominant classroom practices, where instruction is largely procedural, focusing on step-by-step execution of operations that students are expected to replicate. This approach limits opportunities for mathematical reasoning in unfamiliar contexts and negatively affects student motivation in the subject (e.g., Araya-Crisóstomo & Urrutia, 2022; Durán-Vargas et al., 2021; Gamarra-Astuhuaman & Pujay, 2021; Intriago & Naranjo, 2023; OECD, 2021; Rodríguez et al., 2013). Geometry education is particularly impacted by this rigid instructional model (e.g., Padilla-Ortiz, 2024), and in Chile multiple challenges hinder geometry instruction, including restricted classroom time, textbook organization, and traditional teaching methodologies (Aravena-Díaz & Caamaño, 2013; Aravena-Díaz et al., 2016; Henríquez-Rivas et al., 2021).

This article is derived from an undergraduate thesis for obtaining a bachelor's degree in pedagogy in secondary education in mathematics and physics.

✉ yaguilera21@alumnos.utalca.cl ✉ paulasinttia@gmail.com **(*Correspondence)** ✉ jgatica@utalca.cl

Contribution to the literature

- This study expands the existing literature on homothety instruction by examining it from the perspective of students' mathematical work, addressing challenges in teaching and learning processes within the Chilean educational context. It highlights the task redesign and implementation process carried out by a teacher-in-training during her intermediate practicum, emphasizing the importance of training future mathematics teachers in task design and equipping them with necessary research skills for this role.
- The study contributes to the mathematical working space (MWS) theory, specifically regarding emblematic tasks. Given that reference MWS must be adapted, the study reveals that textbooks in Chile often present inconsistencies that may further challenge students' understanding.
- The findings highlight that this approach not only supports progressive learning but also aligns with emerging pedagogical strategies, providing valuable insights for in-service teachers to enhance instruction and assess students' learning processes more effectively.

As a core component of the national mathematics curriculum, geometry is designed to equip students with problem-solving skills while fostering an understanding of space, shapes, and dimensions (Fernández-Nieto, 2018; Quijano, 2022). The curriculum emphasizes the development of reasoning skills, enabling students to analyze their surroundings and engage in rigorous mathematical thinking (Carreño & Cruz, 2016; Ministry of Education [Chile], 2016a). However, a comparison between the first-year secondary school curriculum (Ministry of Education [Chile], 2016b) and the student textbook (Fresno-Ramírez et al., 2023) reveals inconsistencies. While the curriculum aims to promote conceptual understanding, the textbook prioritizes technical applications and formulaic learning, limiting students' deeper engagement with geometric concepts such as homothety (Gómez-Calalán & Andrade-Molina, 2022). Additionally, significant gaps in activity design were identified, particularly in the integration of educational software, with only a single digital tool suggested for the entire lesson.

Recent research has explored didactic approaches for teaching homothety. One such study, *development of geometric thought in high school students when they learn the concept of homothecy* (Labra-Peña & Vanegas-Ortega, 2022), designed a sequence of activities based on Van Hiele's model. The findings underscore the need for further research into how instructional sequence design shapes students' understanding of homothety. Another relevant study by Castro-Cortés et al. (2019) presents a structured sequence of tasks, activities, and situational problems, emphasizing the importance of engaging students in meaningful activities that enhance mathematical skill development. Similarly, *school curriculum discordances: homothety beyond proportionality* (Gómez-Calalán & Andrade-Molina, 2022) and *adidactical situation for a homothetic transformation in the Euclidean plane* (Siles, 2024) critique the limitations of geometry textbooks issued by the Chilean Ministry of Education. These studies employ the theory of didactical situations to propose alternative instructional strategies aimed at improving the teaching and learning of homothety, with an approach that is more meaningful and contextualized in relation to the mathematical object.

The aforementioned studies highlight the need for further research in geometry education, with a particular emphasis on homothety, given the limited number of existing studies on this mathematical concept. Within this framework, teachers play a crucial role in developing students' geometric reasoning skills, underscoring the importance of adopting effective teaching strategies and resources that enrich the learning process (e.g., Espinoza-Vásquez & Verdugo-Hernández 2022; Fernández-Nieto, 2018; Henríquez-Rivas et al., 2021; Intriago & Naranjo, 2023). To achieve this, instructional designs should be systematic, progressively complex, and aimed at deepening students' mathematical understanding in geometry education (Gómez-Chacón et al., 2016). One promising pedagogical approach is the implementation of task-based learning, which provides valuable insights into students' mathematical reasoning and problem-solving processes (Kuzniak, 2022). To enhance this, integrating information and communication technologies (ICT) is crucial, as they support and expand learning opportunities (Ministry of Education [Chile], 2016a; Perdomo-Andrade, 2022). Among the most effective digital tools, GeoGebra stands out in geometry education, offering students a dynamic and interactive platform to explore geometric concepts (García-Cuéllar, 2023; Intriago et al., 2023; Restrepo-Ochoa, 2022).

From the above mentioned, the central research question emerges: How do first-year secondary students respond to a redesigned task-based proposal that fosters mathematical work on homothety? To address this, the study sets forth the following objective: To analyze the mathematical problem-solving process within a task-based approach aimed at fostering mathematical engagement with homothety among first-year secondary students (aged 15-16) at a public high school in the Maule Region of Chile.

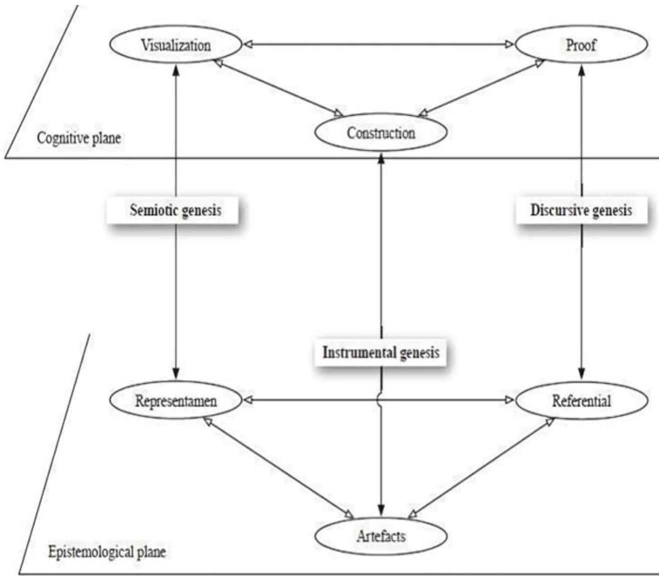


Figure 1. Components of the MWS (Kuzniak et al., 2016, p. 725)

THEORETICAL FRAMEWORK

Mathematical Working Spaces

The MWS theory is a conceptual framework used in educational research to analyze how individuals–teachers, students, or mathematicians–engage in mathematical activity within an educational context (e.g., Espinoza-Vásquez et al., 2025; Kuzniak, 2022; Montoya-Delgadillo & Vivier, 2016; Verdugo-Hernández & Caviedes, 2024) (**Figure 1**). Within this framework, mathematical work is understood as an intellectual and continuously evolving process, structured around three fundamental aspects: the goal, the processes involved, and the mathematical outcomes (Kuzniak, 2022; Kuzniak & Nechache, 2021). Additionally, this theory allows researchers to analyze the actions undertaken by individuals when solving mathematical tasks (Kuzniak, 2022).

The MWS model organizes mathematical activity across two horizontal planes: the epistemological and cognitive dimensions (e.g., Kuzniak, 2011; Kuzniak & Nechache, 2021; Montoya-Delgadillo & Vivier, 2016). The epistemological plane refers to the organization of mathematical knowledge, defining the objects and tools necessary for carrying out mathematical work. The cognitive plane, in contrast, is linked to the student's cognitive engagement, particularly their reasoning processes in mathematical problem-solving. Each of these planes consists of three essential components. Within the epistemological plane, the first component is semiotic representation, known as the representamen, which consists of symbols, signs, and tangible representations. The second component is the referential component, encompassing theoretical foundations such as properties, definitions, and theorems. The third component includes artifacts, which may be material

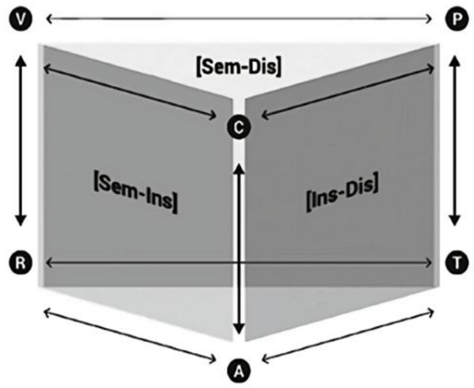


Figure 2. Vertical planes in the MWS (Kuzniak, 2022)

(e.g., drawing tools) or symbolic (e.g., mathematical software) (Kuzniak, 2022). On the cognitive plane, three interrelated components shape students' mathematical activity: visualization, which enables the interpretation of representations and material supports; construction, which depends on the use of artifacts and mathematical techniques; and proof, which is based on the referential component and is fundamental in the validation of mathematical reasoning (e.g., Espinoza-Vásquez & Verdugo-Hernández, 2022; Kuzniak, 2022).

According to MWS theory, mathematical work is a dynamic and progressive process in which epistemological and cognitive components interact. These interactions link both planes through three fundamental types of genesis. The *semiotic genesis* connects the visualization process with semiotic representations, facilitating cognitive comprehension through symbolic decoding and interpretation, transforming signs into operational mathematical objects. The *instrumental genesis* refers to the use of artifacts, enabling them to support construction processes in mathematical work. The *discursive genesis* establishes a link between proof and theoretical references, playing a key role in mathematical reasoning by providing meaning to theorems and mathematical properties (Kuzniak, 2022; Montoya-Delgadillo & Vivier, 2016).

In addition to the components and their associated forms of genesis, Coutat and Richard (2011) introduced the vertical planes, which represent the interaction and circulation between the different genesis types (see **Figure 2**). These vertical planes are classified into three categories: [Sem-Ins], which links semiotic and instrumental genesis; [Ins-Dis], which connects instrumental and discursive genesis; and [Sem-Dis], which circulate between semiotic and discursive genesis (Kuzniak, 2022; Montoya-Delgadillo & Vivier, 2016).

Furthermore, within the MWS theory, three specific types of MWS are defined in the educational context (e.g., Espinoza-Vásquez & Verdugo-Hernández, 2022; Gómez-Chacón et al., 2016). These types allow for a comprehensive study of mathematical work, considering the integration of tools and instruments

(Kuzniak & Nechache, 2016). The three types of MWS include the reference MWS, the suitable MWS, and the personal MWS. The reference MWS establishes a suitable or expected mathematical organization based on formal mathematical criteria. The suitable MWS adapts the reference MWS to make it practical and effective within a specific educational context. The personal MWS is an individual construct that may be developed by a student or teacher in response to mathematical tasks and learning experiences (Kuzniak, 2022).

A mathematical task is considered complete when it meets two key conditions. First, there must be a connection between the epistemological and cognitive planes, ensuring that students use appropriate tools and instruments to address the proposed mathematical situation. Second, the task must allow for smooth transitions between different forms of genesis and vertical planes, incorporating the use of various components, including tools, techniques, and mathematical properties (Kuzniak & Nechache, 2016). Although the MWS theory does not explicitly define the concept of a task, tasks play a critical role in mathematical activity by activating mathematical work and enabling the study of its circulations (Kuzniak, 2011; Montoya-Delgadillo et al., 2014).

Kuzniak (2022) emphasizes that tasks serve as a medium for problem-solving, aligning with Sierpinska's (2004) definition, which was later refined by Nechache (2017). According to this perspective, "*a 'mathematical task' refers to any type of mathematical exercise, question, or problem, with clearly formulated assumptions and questions, that is known to be solvable in a timely manner by students in a well-defined MWS*" (p. 8).

This definition is rooted in the idea that tasks function as a bridge for students to apply techniques and knowledge. Consequently, they are not merely instructional tools but essential elements in shaping students' mathematical work and problem-solving strategies. Moreover, tasks are shaped by the objectives they aim to achieve and the actions required for their execution.

Examining tasks in relation to their implementation allows for a dual analytical approach. First, it focuses on the study of signs, techniques, and theoretical properties that are mobilized during mathematical work. Second, it considers the discrepancies between students' expected mathematical development and their actual performance (Kuzniak, 2022). In this context, Kuzniak and Nechache (2016) introduced the concept of "emblematic tasks", which are characterized by their representativeness of instructional practices. These tasks reflect the pedagogical decisions made by teachers during the teaching process (Henríquez-Rivas et al., 2022). Their validity is supported through analysis and experimentation, offering a comprehensive perspective on mathematical work.

For a task to be considered emblematic, it must meet three conditions simultaneously (Kuzniak, 2022). First, it must be included in the reference MWS, which, in this study, corresponds to the mathematics curriculum framework for first-year secondary education. Second, it must be integrated into the suitable MWS, as defined in the student textbook, teacher's didactic guide, and regular classroom activities. Third, it must support the development of complete mathematical work, facilitating circulation and interaction among components, genesis, and vertical planes of the MWS.

METHOD

Research Characteristics

To analyze students' mathematical work through the implementation of a task-based proposal for secondary education, this study follows a qualitative approach (Denzin & Lincoln, 2012). It is designed as a case study (Stake, 2007) and focuses on a single representative case (Yin, 2018) within a specific school context, allowing for an in-depth evaluation of students' mathematical work with the proposed task. The research was conducted in five key stages:

1. **First stage. Literature review:** A review of existing research (Creswell, 2014) was conducted to identify the characteristics of a task that promotes students' mathematical work on homothety. This included analyzing curriculum documents across different educational levels.
2. **Second stage. Study of mathematical objects:** This stage focused on the reference MWS, examining the mathematics curriculum for first-year secondary education (Ministry of Education [Chile], 2016b). The curriculum was compared with the student textbook proposal (Fresno-Ramírez et al., 2023) to identify a curriculum-based task related to homothety.
3. **Third stage. Task adaptation:** Within the suitable MWS, the selected task was adapted for instructional use, considering insights from the literature review. The adapted task was validated by experts in mathematics education (Galicia-Alarcón et al., 2017) and revised based on expert feedback.
4. **Fourth stage. Classroom implementation:** Within the effective suitable MWS, the adapted task was implemented in four instructional sessions, each lasting 90 minutes.
5. **Fifth stage. Task analysis:** The students' responses from the fourth stage were analyzed using a task analysis protocol (Espinoza-Vásquez & Verdugo-Hernández, 2022) to examine their mathematical work.

Table 1. Protocol used to analyze circulations within the MWS (Espinoza-Vásquez & Verdugo-Hernández, 2022)

Criteria	Component	Descriptor
Semiotic genesis (SG)	Representamen	Establishes relationships between mathematical objects and significant elements.
	Visualization	Interprets and connects mathematical objects with registers of semiotic representation (identification, transformations, and conversions).
Instrumental genesis (IG)	Artifact	Uses material artifacts or symbolic systems.
	Construction	Based on processes triggered by the artifacts used and the associated usage techniques.
Discursive genesis (DG)	Referential	Uses definitions, properties, or theorems.
	Proof	Discursive reasoning relies on various forms of justification, argumentation, or proof.
Vertical plane	[Sem-Ins]	Artifacts are used to construct results under specific conditions or to explore semiotic representations.
	[Ins-Dis]	The proof process is based on experimentation with an artifact or on validating a construction.
	[Sem-Dis]	The validation process of represented objects is coordinated with discursive reasoning to establish proof.

For the purposes of this study, the focus on the second stage, which involves the study of the mathematical object within the reference MWS, and the third stage, which entails the adaptation of the task within the suitable MWS, is addressed briefly. Greater emphasis is placed on the fourth stage, which involves classroom implementation within the effective suitable MWS, and the fifth stage, which pertains to task analysis within the personal MWS of students.

Population and Sample

The sample comprises first-year secondary students from a public high school in the Maule Region of Chile, aged 15 to 16 years. The selection is based on the interest in examining a representative case for each task, aimed at providing deeper insights into students' understanding of the mathematical object.

Instrument: Task Sequence

The task sequence is adapted to the educational level at which it is implemented, specifically first-year secondary education. The sequence spans four instructional sessions, as outlined below:

1. Session 1. Task 1 – Part 1: "Basic notions," including the introduction, and part 2: "Basic notions with concrete materials".
2. Session 2. Task 2 – "Construction with ruler and compass to determine the homothety ratio."
3. Session 3. Task 3 – "Constructing direct and inverse homotheties using GeoGebra."
4. Session 4. Task 4 – "Real-world applications of homothety using GeoGebra applets."

The adaptations of the task sequence were carried out by a pre-service teacher in her fourth year of training, who redesigned, piloted, and validated the tasks in collaboration with experts in didactics and mathematics, following the stages previously described. The research-action practice involved 12 hours of fieldwork, allowing for a gradual immersion in the school environment. This process enabled engagement with different educational levels, identification of existing challenges, and the design and execution of proposals aimed at improving the teaching and learning of mathematics.

Analysis

The study adopts the methodology proposed by Kuzniak and Nechache (2021). This approach consists of describing and analyzing the main actions involved in completing a task by segmenting students' activity into episodes, each comprising a sequence of mathematical actions. After identifying these episodes, an analysis was conducted using a protocol that defines descriptors based on various criteria, which describe the circulation between MWS components (**Table 1**).

RESULTS

Considering the second stage, a review of the first-year secondary mathematics curriculum (Ministry of Education [Chile], 2016b) was conducted. This curriculum includes a unit dedicated to geometry, which establishes the following learning objective (OA8):

> "Evidence understanding of the concept of homothety by relating it to perspective, the function of optical instruments, and the human eye; measuring appropriate segments to determine homothety properties; applying homothety properties to construct objects manually and/or using educational software; and solving real-world and interdisciplinary problems."

As part of the third stage, a homothety-related task was identified in the mathematics textbook by Fresno-Ramírez et al. (2023). While this task introduces homothety, it exhibits a notable disconnect from students' real-world experiences and lacks an explicit connection to their prior mathematical knowledge (Labra-Peña & Vanegas-Ortega, 2022). The example provided features an inverse homothety (see **Figure 3**), which is uncommon in everyday contexts as presented.

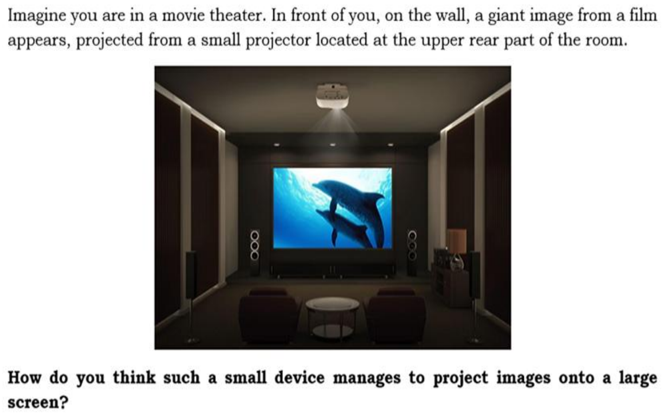
Homotecia

Un pino corresponde a un tipo de árbol con tronco fuerte y rugoso, cuyas hojas son estrechas y parecen agujas. Existen más de 100 especies de pinos y están repartidas por todo el mundo en diferentes continentes. Algunos pinos se encuentran casi extintos y requieren que sean protegidos en parques nacionales para asegurar su bienestar.

Observa la siguiente imagen, y luego responde.

- ¿En qué se parecen los dos pinos?
- ¿Cómo podrías obtener el pino de la derecha a partir del de la izquierda?
- Investiga junto con tus compañeros acerca de si los pinos producen algún efecto negativo o positivo en el medioambiente chileno. Argumenta tu respuesta.

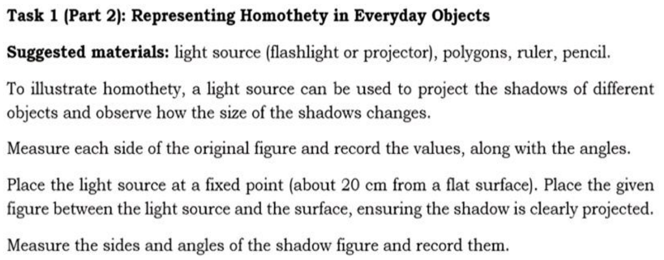
Figure 3. Introduction to the concept of homothety (Fresno-Ramírez et al., 2023, p. 107)



Imagine you are in a movie theater. In front of you, on the wall, a giant image from a film appears, projected from a small projector located at the upper rear part of the room.

How do you think such a small device manages to project images onto a large screen?

Figure 4. Excerpt from the proposed introductory task 1 (Source: Authors' own elaboration)



Task 1 (Part 2): Representing Homothety in Everyday Objects

Suggested materials: light source (flashlight or projector), polygons, ruler, pencil.

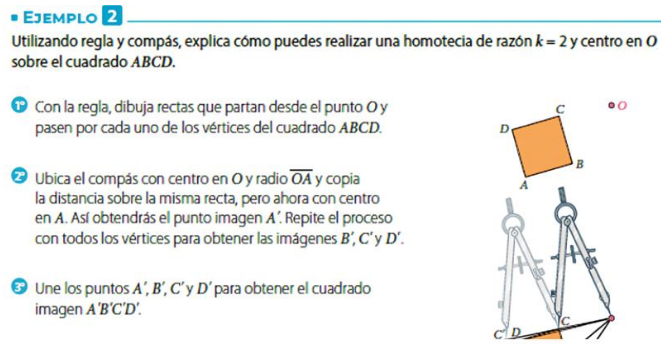
To illustrate homothety, a light source can be used to project the shadows of different objects and observe how the size of the shadows changes.

Measure each side of the original figure and record the values, along with the angles.

Place the light source at a fixed point (about 20 cm from a flat surface). Place the given figure between the light source and the surface, ensuring the shadow is clearly projected.

Measure the sides and angles of the shadow figure and record them.

Figure 5. Task 1 proposal as a practical activity simulating homothety with everyday objects (Source: Authors' own elaboration)

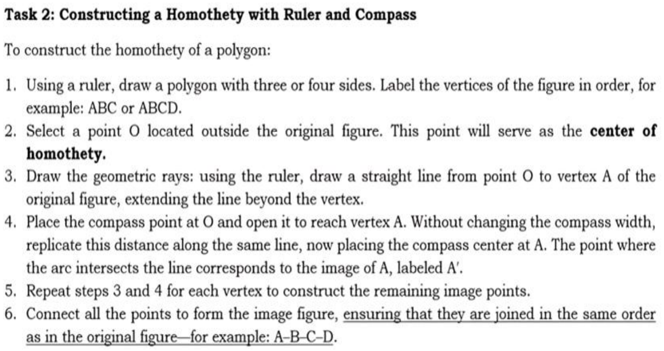


EJEMPLO 2

Utilizando regla y compás, explica cómo puedes realizar una homotecia de razón $k = 2$ y centro en O sobre el cuadrado $ABCD$.

1. Con la regla, dibuja rectas que partan desde el punto O y pasen por cada uno de los vértices del cuadrado $ABCD$.
2. Ubica el compás con centro en O y radio $\overline{OA}$ y copia la distancia sobre la misma recta, pero ahora con centro en A. Así obtendrás el punto imagen A'. Repite el proceso con todos los vértices para obtener las imágenes B', C' y D'.
3. Une los puntos A', B', C' y D' para obtener el cuadrado imagen $A'B'C'D'$.

Figure 6. Excerpt from the student textbook activity on construction with ruler and compass (Fresno-Ramírez et al., 2023, p. 108)

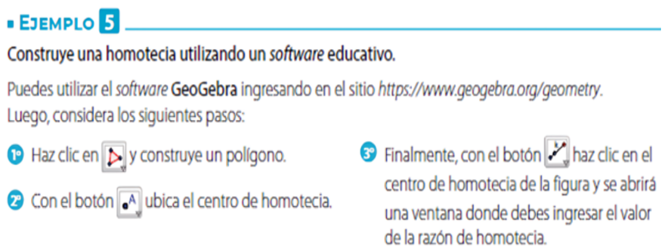


Task 2: Constructing a Homothety with Ruler and Compass

To construct the homothety of a polygon:

1. Using a ruler, draw a polygon with three or four sides. Label the vertices of the figure in order, for example: ABC or ABCD.
2. Select a point O located outside the original figure. This point will serve as the **center of homothety.**
3. Draw the geometric rays: using the ruler, draw a straight line from point O to vertex A of the original figure, extending the line beyond the vertex.
4. Place the compass point at O and open it to reach vertex A. Without changing the compass width, replicate this distance along the same line, now placing the compass center at A. The point where the arc intersects the line corresponds to the image of A, labeled A'.
5. Repeat steps 3 and 4 for each vertex to construct the remaining image points.
6. Connect all the points to form the image figure, ensuring that they are joined in the same order as in the original figure—for example: A-B-C-D.

Figure 7. Task 2 proposal on construction with ruler and compass (Source: Authors' own elaboration)



EJEMPLO 5

Construye una homotecia utilizando un *software* educativo.

Puedes utilizar el *software* **GeoGebra** ingresando en el sitio *https://www.geogebra.org/geometry*. Luego, considera los siguientes pasos:

1. Haz clic en [icon] y construye un polígono.
2. Con el botón [icon] ubica el centro de homotecia.
3. Finalmente, con el botón [icon] haz clic en el centro de homotecia de la figura y se abrirá una ventana donde debes ingresar el valor de la razón de homotecia.

Figure 8. Student textbook activity on construction using the GeoGebra software (Fresno-Ramírez et al., 2023, p. 111)

Additionally, the structure of the introductory example may lead to conceptual confusion, particularly for students already familiar with isometric transformations from previous grade levels.

To create a more effective introductory task, an activity was designed using a projector, a resource available in every classroom where the implementation took place. The proposed task is illustrated in **Figure 4**.

To enhance students' understanding of homothety, a practical activity was introduced, aimed at familiarizing students with the new concept and enabling them to recognize its real-world applications (**Figure 5**).

For task 2, a construction activity using a ruler and compass was designed. The original activity proposed in the textbook (**Figure 6**) was modified by removing the explicit reference to the homothety ratio, shifting the focus towards allowing students to derive and formalize the relevant properties independently. This adaptation resulted in the revised task (**Figure 7**).

For task 3, which required students to construct various homotheties using GeoGebra, the original textbook activity (**Figure 8**) was revised to provide greater structure and focus. The initial task was overly broad in its use of GeoGebra, making it difficult for students to engage meaningfully with the software. The adapted version ensures that students build homotheties with a clear objective, reinforcing the integration of ICT tools in geometry learning.

Figure 8 illustrates the original textbook presentation of homothety, while **Figure 9** presents the revised version, which incorporates a brief instructional manual on GeoGebra. This manual introduces basic functionalities, ensuring that all students develop a common level of proficiency before proceeding with task 3.

Task 3: Determining Homothety Ratios

As we have seen, homothety allows a figure to be enlarged or reduced according to a given ratio. To explore this, complete the following activity using GeoGebra:

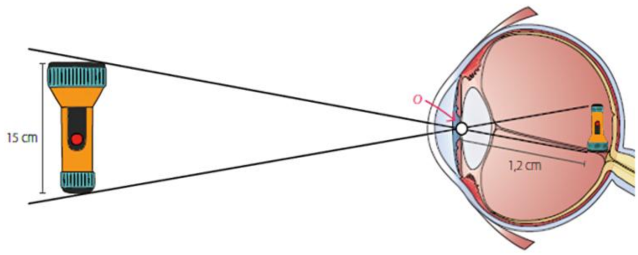
1. Open the GeoGebra software (or app) and draw any triangle or quadrilateral.
2. Construct different homotheties of the figure by **varying the homothety ratio**.

Use the terms: enlargement (dilation) – congruent – reduction (contraction)

Direct Homothety		
If $0 < k < 1$, Example ___________	If $k = 1$	If $k > 1$ Example _________
The image figure corresponds to a ______________ of the original figure.	The image figure is ______________ to the original figure, that is, the image figure coincides with the original.	The image figure corresponds to a ______________ of the original figure.

Figure 9. Task 3 proposal on constructions with GeoGebra and identification of direct and inverse homotheties (Source: Authors' own elaboration)

6. CIENCIAS NATURALES ACTIVIDAD DE PROFUNDIZACIÓN El ojo humano tiene forma parecida a una esfera. Cuando miras algún objeto, este refleja luz que ingresa a nuestros ojos y estos forman una imagen invertida del objeto sobre la retina. **Analiza** la siguiente figura y responde:



a. Decide si la homotecia que se genera al mirar un objeto es directa o inversa. Justifica.

b. ¿Qué signo tiene el factor de homotecia *k*? Justifica

Figure 10. Excerpt from the student textbook activity (Fresno-Ramírez et al., 2023, p. 113)

The final stage of the sequence centered on the design of task 4, which explored real-world applications of homothety, particularly in physics through the use of GeoGebra applets.

The original textbook activity on natural science applications (**Figure 10**) was redesigned into a more structured version titled "image formation in the eye" (**Figure 11**). This adaptation aimed to help students apply prior knowledge, differentiate between various types of homothety, and perform mathematical calculations related to image formation.

Figure 12 shows excerpt 2 from the proposed task 4 on the use of applets and real-life problem-solving.

The fourth and fifth stages outlined in the method section are presented below. To analyze them, the protocol specified in **Table 1** was applied to examine the student's personal MWS in detail.

Implementation and Task Analysis: Effective Suitable MWS and Student's Personal MWS

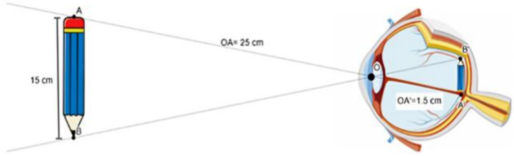
Task 1. Part 1–Basic notions

Description of the work: The teacher [T] interacts dynamically with the students [S] to gradually introduce

Task 4: Applications of Homothety

The following exercises represent applications of homothety in real-life situations. Each is ideally modeled using GeoGebra software.

1. Homothety: Image Formation in the Eye



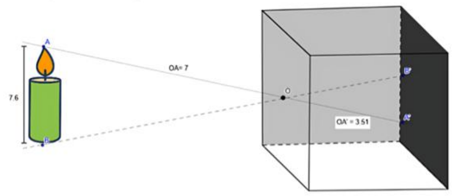
The human eye has a shape similar to a sphere. When we look at an object, it reflects light that enters our eyes and forms an image on the retina.

To better understand image formation, a group of students created a simulation in GeoGebra and asked their classmates to answer the following questions:

a. Is the homothety generated when looking at an object direct or inverse? *Justify your answer.*

Figure 11. Excerpt 1 from the proposed task 4 on the use of applets and real-life problem-solving (Source: Authors' own elaboration)

2. Homothety: Functioning of the Pinhole Camera



The pinhole camera is one of the oldest inventions related to optics and was a precursor to modern photographic cameras. This device consists of a box with a small hole on one side. Light passing through this hole projects an image of the external object onto the opposite surface (the camera's "screen"). This phenomenon occurs due to the refraction of light.

Julio and Ana are studying applications of homothety to real-world and everyday situations in their mathematics class. To support this, the teacher presents the following situation in GeoGebra:

In a pinhole camera, a candle measuring 7.6 cm in height is placed 7 cm from the hole. The distance from the center of homothety to one of the points of the homothetic image is 3.51 cm.

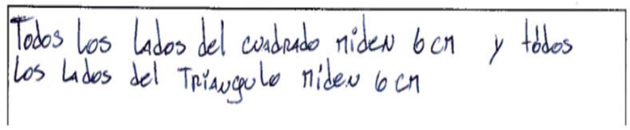
a. Based on the value of k, *how is the orientation of the projected image affected?*

Figure 12. Excerpt 2 from the proposed task 4 on the use of applets and real-life problem-solving (Source: Authors' own elaboration)

the concept of homothety, as seen in the following excerpt [1-4]:

> 1 T: Imagine we are in a movie theater. If you have never been to one, picture yourself here in the classroom, watching a projection on the board. In front of you, on the wall, a giant image of a movie appears, projected from a small projector located at the upper back of the room.
>
> 2 T: Now, how do you think such a small device can project these images onto a large screen? Write down your ideas in your guide.
>
> 3 T: How do you think it achieves that projection?
>
> 4 S1: With light?
>
> 5 T: Okay, write down your ideas.

Later in the lesson, students are encouraged to identify the same mathematical object in familiar devices or instruments, as illustrated in the following interaction [6-15]:



Transcription of the student's response:
"All sides of the square measure 6 cm, and all sides of the triangle measure 6 cm."

Figure 13. Type of register used by the student (Source: Authors' own elaboration)

> 6 T: What other devices or instruments do you think use the same principle to enlarge or reduce images? There is one you have right at your fingertips.
>
> 7 S2: A mobile phone.
>
> 8 T: Can you explain how a phone does it? How do you recognize that it uses homothety?
>
> 9 S2: With photos.
>
> 10 T: How with photos?
>
> 11 S2: When taking a photo, it reduces the size of the large space we want to capture.
>
> 12 T: And it reduces its size. Good.
>
> 13 T: There's another function as well.
>
> 14 S3: Zoom.
>
> 15 T: Very good.

Theoretical analysis: This activity activates the referential component, as the teacher attempts to elicit students' prior knowledge and real-world experiences before introducing the first homothety task. Visualization is also integrated, enabling students to relate the mathematical object to different representations. This section reflects the effective suitable MWS, as the teacher dedicates an entire session to introducing basic homothety concepts due to it being a new topic. The teacher-student interactions are emphasized, fostering an engaging learning environment. The teacher's approach to questioning and guiding students' responses promotes active participation, as evidenced in the exchange (e.g., excerpts [6-15]).

Task 1: Part 2–Representing homothety with everyday objects

Identification of key work episodes: In part 2, four key work episodes [E] were identified:

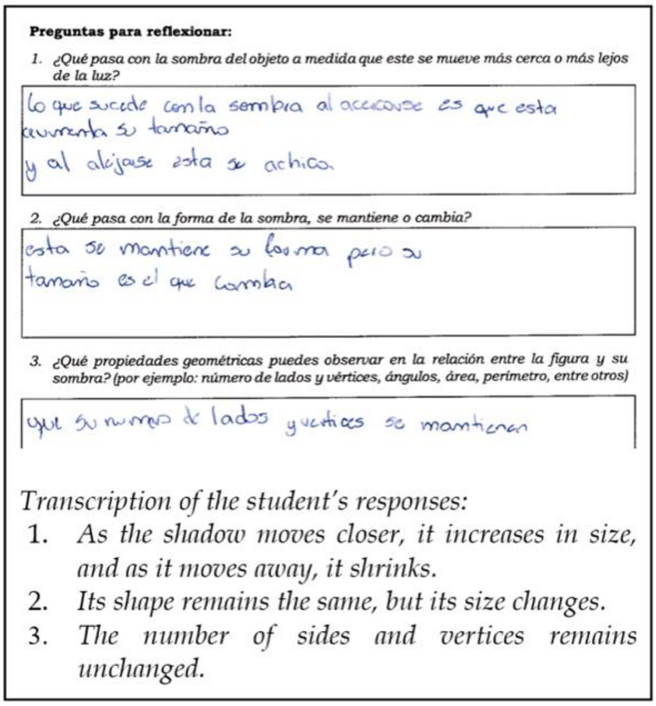
Figure 14. Simulation of an enlargement by moving the object closer to the light source (mobile phone) (Source: Authors' own elaboration)

1. E1: Measuring the side lengths of provided figures.
2. E2: Setting up the system with a light source.
3. E3: Manipulating the system by moving the figure closer to or further from the light source.
4. E4: Formalizing observations and conclusions.

In E1, students measure the side lengths of the provided figures, which include equilateral triangles and squares with 6 cm-long sides. They use various registers of representation, with natural language being the most prevalent, as shown in **Figure 13**.

In E2, students construct a setup to visualize homothety using concrete materials, positioning the light source parallel to the table surface and placing the object between them. This leads to E3, where they adjust the object's distance from the light source (**Figure 14**). Upon completing the practical exercise, they proceed to E4, where they respond to related questions (**Figure 15**). To monitor student progress during E4, the following interaction took place:

> 1 T: Here's another question. What happens to the shape of the shadow? Does it remain the same, or does it change?
>
> 2 S4: It changes.
>
> 3 T: Pay attention. Does the shape of the shadow change?
>
> 4 S4: No, only its size.
>
> 5 T: Good. If you move it closer to the light source, does it stop being a square?
>
> 6 S4: No.


Preguntas para reflexionar:

1. ¿Qué pasa con la sombra del objeto a medida que este se mueve más cerca o más lejos de la luz?

lo que sucede con la sombra al acercarse es que esta aumenta su tamaño
y al alejarse esta se achica

2. ¿Qué pasa con la forma de la sombra, se mantiene o cambia?

esta se mantiene su forma pero su
tamaño es el que cambia

3. ¿Qué propiedades geométricas puedes observar en la relación entre la figura y su sombra? (por ejemplo: número de lados y vértices, ángulos, área, perímetro, entre otros)

que su numero de lados y vertices se mantienen

Transcription of the student's responses:

1. *As the shadow moves closer, it increases in size, and as it moves away, it shrinks.*
2. *Its shape remains the same, but its size changes.*
3. *The number of sides and vertices remains unchanged.*

Figure 15. Example of a student's responses regarding the practical activity (Source: Authors' own elaboration)

7 T: And if you move it farther away, does it stop being a square?

8 S4: No.

Theoretical analysis: According to the theoretical framework, this task activates the referential component, including homothety definitions and properties, as well as segment measurements (E1). Additionally, students use non-mathematical artifacts, such as a light source and concrete materials (E2), which enables construction-based learning (E3). This process activates the instrumental genesis. Furthermore, visualization plays a crucial role, as students engage in a hands-on activity to represent inverse homothety using everyday objects. In E4, students incorporate prior knowledge, such as the geometric properties of shapes, reinforcing referential understanding.

This section illustrates the student's personal MWS, which evolves as they interact with the new content. Throughout their work, they transition across various genesis. The [Sem-Ins] plane is activated, as non-mathematical artifacts, like the light source, facilitate construction and visualization.

Task 2. Constructing homothety with a ruler and compass to determine the homothety ratio

Identification of key work episodes: For this task, seven key work episodes [E] were identified:

1. E1: Drawing a polygon and marking its vertices.
2. E2: Marking the homothety center (point O).
3. E3: Drawing geometric rays.

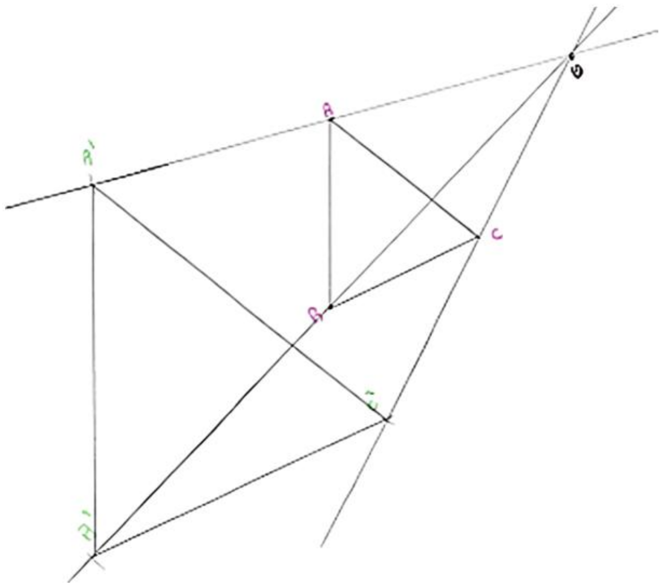


Figure 16. Construction of a direct homothety by student 1 (Source: Authors' own elaboration)

4. E4: Drawing homothetic points using a compass.
5. E5: Connecting homothetic points to form the resulting figure.
6. E6: Measuring the center-to-vertex distance to determine the homothety ratio.
7. E7: Formalizing results.

In E1, students were required to draw a polygon of their choice to serve as the original figure, with triangles and quadrilaterals being the most commonly selected. The vertices were marked in a systematic order. In E2, students identified and marked the homothety center on the designated plane, leading to E3, where they drew geometric rays passing through point O and each vertex of the original figure. Subsequently, in E4, using a compass, students proceeded to copy the center-to-vertex length onto the previously drawn ray. At this stage, students showed two distinct approaches to completing the activity, resulting in different homothetic constructions. To obtain the image figure, students transitioned to E5, where they connected each of the homothetic points, thereby marking the new vertices (see **Figure 16** and **Figure 17**).

Once the construction activity was completed in E6, students measured the distances from the center to each vertex and recorded the obtained values to calculate the ratio between the homothetic and original lengths (see **Figure 18** and **Figure 19**). To guide students in determining the homothety ratio, the following interaction between the teacher [T] and students [S] took place:

1 T: In line with our objective, we are going to determine the homothety ratio. The guide states that we need to calculate a quotient. Do you remember what a quotient is?

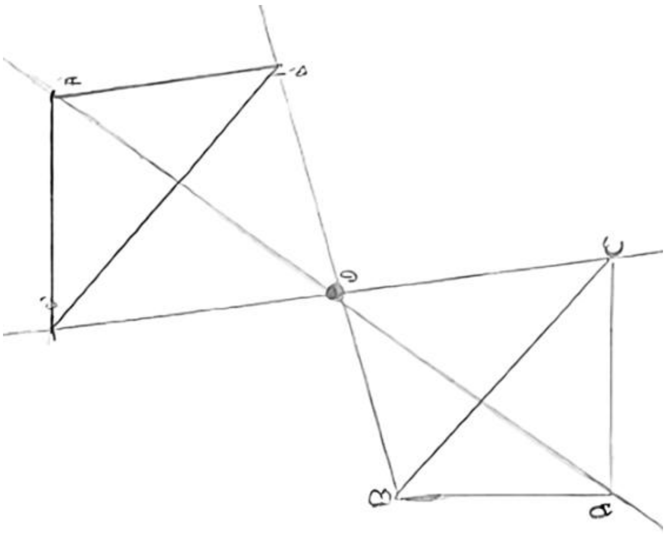


Figure 17. Construction of an inverse homothety by student 2 (Source: Authors' own elaboration)

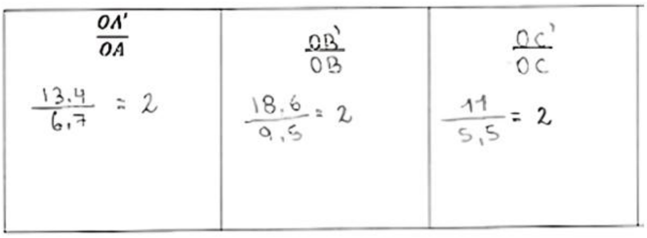


Figure 18. Response of student 1 when calculating the ratio of direct homothety (Source: Authors' own elaboration)

Figure 19. Response of student 2 when calculating the ratio of inverse homothety (Source: Authors' own elaboration)

2 S5: We perform a division.

3 T: Very good.

4 T: Now, let's measure the distance between point O and vertex A. Check your constructions and record the measurements in your guides.

5 T: Now that you have measured this distance, you should repeat the process for each vertex in both the image figure and the original figure.

The above-mentioned process led to E7, in which students formalized their conclusions by responding to guiding questions (see Figure 20 and Figure 21).

Theoretical analysis: Students engage in the concept of polygons, which activates the referential component

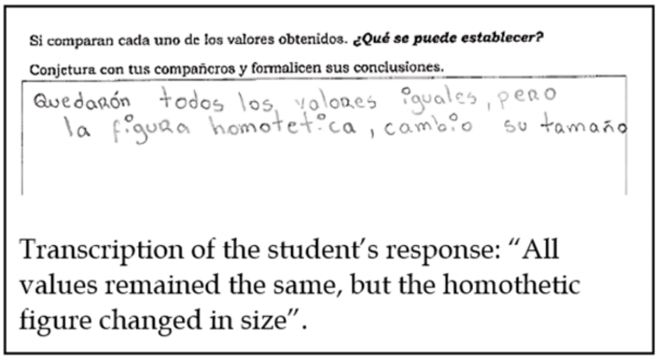


Figure 20. Student 1's response when formalizing results and conclusions (Source: Authors' own elaboration)

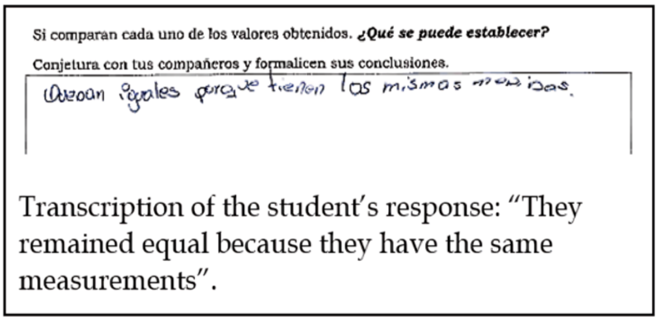


Figure 21. Student 2's response when calculating the ratio of inverse homothety (Source: Authors' own elaboration)

by mobilizing prior knowledge of geometric figures and their properties (E1).

As they progress in the task, students activate the representamen by identifying and constructing the homothety center as a point on the plane (E2).

The use of artifacts plays a central role in the construction process, with the ruler and compass being essential tools for drawing geometric rays (E3) and marking homothetic points (E4), thereby activating the instrumental genesis. This episode reflects a deeper level of conceptual understanding, as the visualization of the figure–by connecting the homothetic points–facilitates the construction of the image figure, activating the semiotic genesis (E5).

Following the construction of the homothety, the proof process is observed in line with Balacheff's (1987). As students measure segment lengths and identify regularities and a consistent homothety ratio (E6), they confirm their constructions by applying geometric properties (E7), thereby activating the discursive genesis.

The previous analysis highlights the progressive activation of different genesis, allowing for transitions between vertical planes. For instance, the proof of the construction activates the [Ins-Dis] plane. The validation of represented objects, using discursive reasoning, engages the [Sem-Dis] plane. Finally, the use of artifacts in the construction process under specific conditions facilitates circulation within the [Sem-Ins] plane.

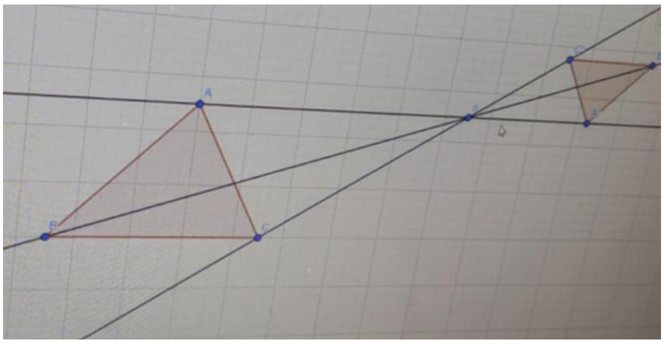


Figure 22. Student 1's response on the construction in GeoGebra (Source: Authors' own elaboration)

Homotecia directa		
Si 0 < k < 1, Ejemplo 0.5	Si k = 1	Si k > 1 Ejemplo 5
La figura imagen corresponde a una Reducción de la figura original.	La figura imagen es Congruente con la figura original, es decir, la figura imagen coincide con la figura original.	La figura imagen corresponde a una Ampliación de la figura original.

Homotecia inversa		
Si -1< k < 0 Ejemplo -0.5	Si k = -1	Si k < -1 Ejemplo -2
La figura imagen corresponde a una Reducción de la figura original y se invierte su sentido.	La figura imagen es Congruente con la figura original y se invierte su sentido.	La figura imagen corresponde a una ampliación de la figura original y se invierte su sentido.

Figure 23. Student 1's response when identifying direct and inverse homotheties (Source: Authors' own elaboration)

Task 3: Construction of direct and inverse homotheties in GeoGebra

Identification of key work episodes: In this task, five key work episodes [E] were observed:

1. E1: Drawing any polygon.
2. E2: Marking the homothety center.
3. E3: Drawing geometric rays.
4. E4: Constructing a homothety with an arbitrary ratio.
5. E5: Varying the homothety ratio to distinguish between direct and inverse homotheties and identifying their properties.

In E1, students were required to construct a polygon of their choice using GeoGebra, drawing on prior knowledge from the previous task. In E2 and E3, they marked a random point on the plane as the homothety center and then drew geometric rays through it. Subsequently, in E4, students used GeoGebra's built-in homothety function to construct a homothety with an arbitrary ratio (see **Figure 22**). This led to E5, where they varied the ratio values to generate different homotheties, enabling them to identify key properties (see **Figure 23**) and answer associated questions (see **Figure 24**).

1. **Si al aplicar una homotecia, la figura imagen es congruente con la figura original y quedan en el mismo lado respecto del centro O,** *¿Cuál es el valor de k?*

El valor de k es 1.

2. **Si al aplicar una homotecia, la figura imagen es congruente con la figura original, pero se invierte su sentido,** *¿Cuál es el valor de k?*

el valor es -1

3. **Si el valor de la razón de homotecia cumple con k < 0,** *¿Siempre se obtiene una reducción de la figura? Escribe un ejemplo.*

se obtiene cuando k esta entre -1 y 0

Transcription of the student's responses:

1. "The value of k is 1"
2. "The value of k is -1"
3. "It is obtained when k is between -1 and 0"

Figure 24. Formalization of student 1's responses (Source: Authors' own elaboration)

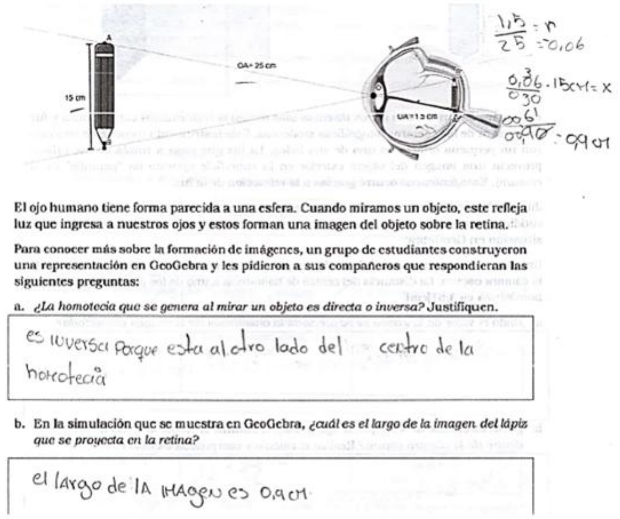
Theoretical analysis: In E1, students utilized digital drawing tools in GeoGebra, activating the instrumental genesis while constructing a geometric representation. The representamen was also engaged, as students related the mathematical object to its geometric register through their polygon construction. In E2 and E3, students identified and marked the homothety center, then drew the necessary rays. These actions further mobilized representamen and visualization, leading to the activation of semiotic genesis. In E4, GeoGebra was used as an artifact to construct direct and inverse homotheties, facilitating a transition to instrumental genesis. In E5, as students modified the homothety ratio, they activated the referential component, requiring an understanding of geometric properties to correctly interpret and identify homothetic transformations.

The semiotic genesis was evident in how students engaged with the software's visualization tools, which acted as a bridge between mathematical objects and their representations. Likewise, the instrumental genesis was mobilized as students manipulated the GeoGebra interface to complete various constructions. These two genesis interacted through the vertical plane [Sem-Ins], where artifacts were used under specific conditions to construct homotheties.

Task 4: Applications of homothety in real-world situations

Identification of key work episodes: For this task, four key work episodes [E] were observed:

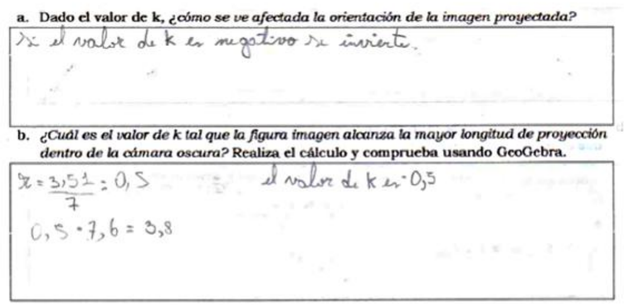
1. E1: Manipulation of GeoGebra applets.



Transcription of the student's responses:

a. "It is inverse because it is on the opposite side of the homothety center."
b. "The length of the image is 0.9 cm."

Figure 25. Student 2's response to part 1 of the task (Source: Authors' own elaboration)



Transcription of the student's responses:

a. "If the value of K is negative, it inverts."
b. k = 3.51/7 = 0.5 "The value of k is -0.5"
0.5 * 7.6 = 3.8.

Figure 26. Student 1's response to part 2 of the task (Source: Authors' own elaboration)

2. E2: Recording explicit data.
3. E3: Performing required calculations.
4. E4: Formalizing the obtained results.

In E1, students explored two problem scenarios involving real-world applications of homothety, manipulating pre-designed GeoGebra applets. This led to E2, where students recorded essential data in their work guides. Furthermore, in E3, they calculated the homothety ratio and the length of homothetic segments. Finally, in E4, students formalized their findings, allowing them to answer key questions (see **Figure 25** and **Figure 26**).

Theoretical analysis: In E1, visualization was emphasized, enabling students to interpret homothety through concrete examples and explore its function using graphical representations. The ability to manipulate the artifact activated semiotic genesis, as students linked the mathematical object to its semiotic representation within the digital tool. In E2, students engaged in data recording, reinforcing the representamen component by linking mathematical objects to meaningful elements in the construction process. In E3, students applied homothety definitions and properties, activating the referential component. This knowledge enabled them to perform calculations and transition into E4, where they formalized their results and conclusions.

Regarding the activated genesis and vertical planes, the instrumental genesis was mobilized as students manipulated the digital tool to complete their constructions and analyze homothety's function. The transition between artifacts and their semiotic representations supported the activation of the [Sem-Ins] plane.

DISCUSSION

The results indicate how a sequence of tasks fosters mathematical work among first-year secondary students in the study of homothety, addressing a key need identified by Labra-Peña and Vanegas-Ortega (2022), regarding the importance of exploring didactical strategies that deepen students' conceptual understanding. The task sequence is designed to overcome limitations in geometry instruction, particularly those stemming from textbook-based approaches that emphasize rote memorization (Gómez-Calalán & Andrade-Molina, 2022). Instead, this approach prioritizes students' active knowledge construction through their own productions and task progression.

From the perspective of the MWS framework, which enabled the analysis of students' mathematical activity throughout the task sequence, a progressive activation of the components, geneses, and planes that constitute their personal MWS was identified. To provide an overview of this progression, **Table 2** summarizes the task-by-task analysis.

Task 1 introduced students to the concept of homothety through the use of concrete materials, fostering teacher-student interaction to support conceptual acquisition. This initial stage of the sequence predominantly activated the components of visualization and artifact use, enabling students to establish connections between the mathematical object and various forms of representation. The interaction between these components strengthened the instrumental plane, which became dominant due to the involvement of non-mathematical artifacts. This task

Table 2. Summary of activated components, genesis, and planes in the student's personal MWS

Task	Topic	Activated components	Activated genesis	Activated planes
T. 1 (Introductory)	Basic notions using concrete materials	Representamen, visualization, artifacts, construction, & referential	Semiotic instrumental	[Sem-Ins]
T. 2	Construction with ruler and compass to determine the homothety ratio	Representamen, visualization, artifacts, construction, referential, & proof	Semiotic instrumental discursive	[Ins-Dis] [Sem-Dis] [Sem-Ins]
T. 3	Homothety constructions in GeoGebra (direct and inverse)	Representamen, visualization, artifacts, construction, & referential	Semiotic instrumental	[Sem-Ins]
T. 4	Real-world applications of homothety using GeoGebra applets	Representamen, visualization, artifacts, construction, & referential	Semiotic instrumental	[Sem-Ins]

thus lays the conceptual foundation for engaging with the mathematical object under study.

Task 2 was one of the most complex in the sequence, as it incorporated mathematical artifacts such as the ruler and compass to explore complex concepts related to homothety construction. This task privileged instrumental and discursive genesis, aligning with Kuzniak (2022), who argues that these components are essential for validating mathematical constructions. The task also allowed students to engage in Balacheff's (1987, 2000) proof model, where they articulate conclusions and validate them through calculations–for instance, by determining the homothety ratio for all segments.

Task 3 and task 4 introduced a digital tool, GeoGebra, as an artifact that contributes to the activation of students' personal MWS. The integration of this software not only activated semiotic genesis but also facilitated connections between visualization and referential components within real-world contexts, aligning with the claims of Restrepo-Ochoa (2022) and García-Cuéllar (2023) regarding the value of this tool for exploring geometric concepts dynamically. The use of GeoGebra also mobilized instrumental genesis, thereby activating the semiotic-instrumental plane. These tasks enabled students to apply and consolidate their understanding of the definitions and properties of homothety.

The task analysis and results confirm that the implemented sequence meets the third condition for an emblematic task as proposed by Kuzniak and Nechache (2016): the task fosters complete mathematical work for students. The evidence collected during the implementation illustrates the activation and circulation among the various components, geneses, and planes that constitute students' personal MWS.

CONCLUSIONS

The designed and implemented task sequence aligns with the definition of an emblematic task, as proposed by Kuzniak and Nechache (2016). It is representative of classroom instruction within this specific educational context, fulfilling the three necessary conditions: presence in the reference MWS, as it is included in the mathematics curriculum program; integration into the suitable MWS, as it appears in the teacher's guide and student's textbook; and facilitation of complete mathematical work, achieved progressively throughout the sequence.

Since tasks serve as a medium for studying mathematical work, the results directly address the research question regarding how do first-year secondary students respond to a redesigned task-based proposal that fosters mathematical work on homothety? Furthermore, the analysis of students' mathematical solutions provides valuable information for teachers, as it reveals their level of understanding of the content addressed and helps identify difficulties or errors that may arise, insights that are essential for adjusting instructional strategies and feedback.

Overall, the student's personal MWS is significantly activated, allowing them to navigate different components to organize their mathematical work according to their understanding and skill set. However, initial limitations were observed due to students' challenges in operating certain artifacts, which initially hindered their ability to construct mathematical objects. Nevertheless, these challenges provided an opportunity for students to adapt to their mathematical work and successfully engage with the proposed learning situations. This study aims to contribute to the teaching of homothety and serve as a foundation for designing future task sequences in geometry and other mathematical domains. The findings highlight the potential for improving instructional approaches in mathematics textbooks, fostering a more student-centered learning experience.

One limitation of this study is time constraints, as the task sequence was implemented during the second semester of the Chilean school year. This period is affected by external factors in educational institutions. For future research, a broader case study is recommended to analyze students' mathematical work progression before and after the task sequence implementation. This would allow for a deeper understanding of how students engage with new mathematical concepts and how their learning process evolves through task-based instruction.

Author contributions: YAM, PVH, & JG: conceptualization, methodology, data analysis, writing, review, and editing; **YAM:** draft writing of the paper; & **PVH:** funding acquisition. All authors agreed with the results and conclusions.

Funding: The corresponding author was funded by the National Research and Development Agency of Chile through the National Fund for Scientific and Technological Development for Initiation 2023, Project Folio 11230240.

Ethical statement: The authors stated that the study was approved by the Reaccredited Scientific Ethics Committee of the Maule Regional Health Ministry with approval number 1976/28.09.22. Written informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Agency for Quality Education (Chile). (2020). TIMSS 2019: International trends in mathematics and science study. *Agencia Educacion*. https://archivos.agenciaeducacion.cl/Presentacion_TIMSS_2019_version_oficial.pdf

Aravena-Díaz, M., & Caamaño, C. (2013). Levels of geometric reasoning in students of statal schools of Maule Region, Talca, Chile. *Revista Latinoamericana de Investigación en Matemática Educativa, 16*(2), 139-178. https://doi.org/10.12802/relime.13.1621

Aravena-Díaz, M., Gutiérrez M., & Jaime, A. (2016). Study of van Hiele levels of reasoning in students from vulnerable secondary schools in Chile. *Enseñanza de las Ciencias, 34*(1), 107-128. https://doi.org/10.5565/rev/ensciencias.1664

Araya-Crisóstomo, S., & Urrutia, M. (2022). Use of participatory methodologies in pedagogical practices of school system. *Pensamiento Educativo. Revista de Investigación Educacional Latinoamericana, 59*(2), 1-16. https://doi.org/10.7764/pel.59.2.2022.9

Balacheff, N. (1987). Processus de prevue et situations de validation. *Educational Studies in Mathematics, 18*(2), 147-176. https://doi.org/10.1007/BF00314724

Balacheff, N. (2000). Procesos de prueba en los alumnos de matemáticas. Una empresa docente [Testing processes in mathematics students. A teaching company]. *HAL Open Science*. https://hal.science/hal-00520133/document

Carreño, X., & Cruz, X. (2016). *Geometría* [Geometry]. McGraw-Hill.

Castro-Cortés, A. M., Jaramillo Riascos, J. C., & Obregón Valencia, I. A. (2019). *Una aproximación al concepto de homotecia a partir de la noción de proporcionalidad geométrica en séptimo grado* [An approach to the concept of homothecy based on the notion of geometric proportionality in the seventh degree] [Undergraduate thesis, Universidad del Valle].

Coutat, S., & Richard, P. R. (2011). Les figures dynamiques dans un espace de travail mathématique pour l'apprentissage des propriétés mathématiques [Dynamic figures in a mathematical workspace for learning mathematical properties]. *Annales de Didactique et de Sciences Cognitives, 16*, 97-126.

Creswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approach*. SAGE.

Denzin, N., & Lincoln, Y. (2012). *Manual de investigación cualitativa* [Qualitative research manual]. Gedisa.

Durán-Vargas, C., Mora Cid, F., Smith Ramírez, A., & Vera Vera, D. (2021). *Experiencias de profesores que enseñan matemática a partir de la utilización de las metodologías activas y tradicionales en la zona sur de Chile* [Experiences of teachers teaching mathematics using active and traditional methodologies in southern Chile] [Undergraduate thesis, Universidad Católica de la Santísima Concepción].

Espinoza-Vásquez, G., Henríquez-Rivas, C., Climent, N., Ponce, R., & Verdugo-Hernández, P. (2025). Teaching Thales's theorem: Relations between suitable mathematical working spaces and specialised knowledge. *Educational Studies in Mathematics, 118*, 271-293. https://doi.org/10.1007/s10649-024-10367-9

Espinoza-Vásquez, G., & Verdugo-Hernández, P. (2022). Representations of the function during teaching A look from teacher's specialized knowledge and mathematical work. *International Humanities Review, 11*(6), 1-18. https://doi.org/10.37467/revhuman.v11.4082

Fernández-Nieto, E. L. (2018). La geometría para la vida y su enseñanza [Geometry for life and its teaching]. *Aibi Revista de Investigación, Administración e Ingeniería, 6*(2), 33-61. https://doi.org/10.15649/2346030X.475

Fresno-Ramírez, C., Torres-Jeldes, C., & Ávila-Hidalgo, J. (2023). *Texto del estudiante 1° medio matemática* [Student text 1st year mathematics]. Santillana del Pacífico S. A.

Galicia-Alarcón, L. A., Balderrama Trápaga, J. A., & Edel Navarro, R. (2017). Content validity by experts judgment: Proposal for a virtual tool. *Apertura, 9*(2), 42-53. https://doi.org/10.32870/Ap.v9n2.993

Gamarra-Astuhuaman, G., & Pujay, O. E. (2021). Problem solving, skills and academic performance when teaching mathematics. *Revista Educación, 45*(1), 176-189. https://doi.org/10.15517/revedu.v45i1.41237

Gaona J. (2022). Task design in an online assessment system, a view from the theory of mathematical working spaces. *PädiUAQ, 5*(10), Article e202202. https://revistas.uaq.mx/index.php/padi/article/view/652

García-Cuéllar, D. J. (2023). Teaching and learning of geometry with GeoGebra. *REAMEC-Revista Amazónica de Educación en Ciencias y Matemática, 11*(1), Article e23118. https://doi.org/10.26571/reamec.v11i1.16880

Gómez-Calalán, J., & Andrade-Molina, M. (2022). School curriculum discordances: Homothety beyond proportionality. *Revista Chilena de Educación Matemática, 14*(1), 31-42. https://doi.org/10.46219/rechiem.v14i1.105

Gómez-Chacón, I. M., Kuzniak, A., & Vivier, L. (2016). The teacher's role from the perspective of mathematical working spaces. *Bolema: Boletim de Educação Matemática, 30*(54), 1-22. https://doi.org/10.1590/1980-4415v30n54A01

Henríquez-Rivas, C., Kuzniak, A., & Masselin, B. (2022). The iodine or suitable MWS as an essential transitional stage between personal and reference mathematical work. In A. Kuzniak, E. Montoya-Delgadillo, & R. Philippe (Eds.), *Mathematical work in educational context* (pp. 121-146). Springer. https://doi.org/10.1007/978-3-030-90850-8_6

Henríquez-Rivas, C., Ponce, R., Carrillo Yáñez, J., Climent, N., & Espinoza-Vásquez, G. (2021). Mathematical work of a teacher based on tasks and examples proposed for teaching. *Enseñanza de las Ciencias, 39*(2), 123-142. https://doi.org/10.5565/rev/ensciencias.3210

Intriago, S. M., & Naranjo, C. A. (2023). Learning mathematics in basic general education students. *RECIMUNDO: Revista Científica de la Investigación y el Conocimiento, 7*(1), 640-653. https://doi.org/10.26820/recimundo/7.(1).enero.2023.640-653

Intriago, Y., Vergara, J., & López, R. (2023). Use of didactic resources, from learning analytics in the transformations of mathematics teaching in plane geometry. *MQRInvestigar, 7*(3), 2278-2296. https://doi.org/10.56048/MQR20225.7.3.2023.2278-2296

Kuzniak, A. (2011). L'espace de travail mathématique et ses genèses [The mathematical workspace and its genesis]. *Annales de Didactique et de Sciences Cognitives, 16*, 9-24.

Kuzniak, A. (2022). The theory of mathematical working spaces–Theoretical characteristics. In A. Kuzniak, E. Montoya-Delgadillo, & R. Philippe (Eds.), *Mathematical work in educational context* (pp. 3-31). Springer. https://doi.org/10.1007/978-3-030-90850-8_1

Kuzniak, A., & Nechache, A. (2016). Tâches emblématiques dans l'étude des ETM idoines et personnels: Existence et usages [Emblematic tasks in the study of suitable and personal ETMs: Existence and uses]. In *Proceedings of the 5th Symposium on Mathematical Workspace*.

Kuzniak, A., & Nechache, A. (2021). On forms of geometric work: A study with pre-service teachers based on the theory of mathematical working spaces. *Educational Studies in Mathematics, 106*, 271-289. https://doi.org/10.1007/s10649-020-10011-2

Labra-Peña, J. A., & Vanegas-Ortega, C. M. (2022). Development of geometric thought in high school students, when they learn the concept of homothecy. *Revista Latinoamericana de Investigación en Matemática Educativa, 25*(1), 93-120. https://doi.org/10.12802/relime.22.2514

Ministry of Education (Chile). (2016a). Bases curriculares 7° básico a 2° medio [Curricular guidelines 7th grade to 10th grade]. *Gobierno de Chile*. https://www.curriculumnacional.cl/614/articles-37136_bases.pdf

Ministry of Education (Chile). (2016b). Programa de estudio primero medio matemática [Mathematics study program for first year of secondary education]. *Biblioteca Digital* Mineduc. https://www.curriculumnacional.cl/614/articles-34359_programa.pdf

Montoya-Delgadillo, E., & Vivier, L. (2016). Mathematical working space and paradigms as an analysis tool for the teaching and learning of analysis. *ZDM Mathematics Education, 48*, 739-754 https://doi.org/10.1007/s11858-016-0777-9

Montoya-Delgadillo, E., Mena-Lorca, A., & Mena-Lorca, J. (2014). Circulations and genesis in the mathematical work space. *Revista Latinoamericana de Investigación en Matemática Educativa, 17*(4(I)), 181-197. https://doi.org/10.12802/relime.13.1749

Nechache, A. (2017). La catégorisation des tâches et du travailleur-sujet: Un outil méthodologique pour l'étude du travail mathématique dans le domaine des probabilités [Task and worker-subject categorization: A methodological tool for the study of mathematical work in the field of probability]. *Annales de Didactique et de Sciences Cognitives, 19*, 67-90. https://doi.org/10.4000/adsc.709

OECD. (2021). PISA 2021 mathematics framework (draft). *Organization for Economic Co-operation and Development*. https://pisa2022-maths.oecd.org/files/PISA%202021%20Mathematics%20Framework%20Draft.pdf

OECD. (2023). PISA 2022 results (volume I & II)–Country notes: Chile. *Organization for Economic Co-operation and Development*. https://www.oecd.org/en/publications/pisa-2022-results-volume-i-and-ii-country-notes_ed6fbcc5-en/chile_d038b73d-en.html

Padilla-Ortiz, Y. N. (2024). *La gamificación como estrategia de aprendizaje de geometría en estudiantes de octavo año de educación general básica* [Gamification as a geometry learning strategy for eighth-grade students of basic general education] [Undergraduate thesis, Universidad Nacional de Chimborazo].

Perdomo-Andrade, I. (2022). Review on the use of ICTs in science. *Revista Latinoamericana de Educación Científica, Crítica y Emancipadora-LadECiN, 1*(2), 01-08. https://doi.org/10.5281/zenodo.8076344

Quijano, M. (2022). *Enseñanza de la geometría: Análisis de las praxeologías que se estudian en la escuela secundaria* [Teaching geometry: Analysis of the praxeologies studied in secondary school] [PhD thesis, Universidad Nacional del Centro de la Providencia de Buenos Aires].

Restrepo-Ochoa, J. F. (2022). *Análisis del proceso de aprendizaje de la proporcionalidad, la semejanza y la homotecia usando el modelo de van Hiele, la visualización matemática y el software GeoGebra*. [Analysis of the learning process of proportionality, similarity, and homothety using the van Hiele model, mathematical visualization, and GeoGebra software] [Master's thesis, Universidad Autónoma de Bucaramanga].

Rodríguez, M. B., Carreño, X., Muñoz, V., Ochsenius, H., Mahías, P., & Bosch, A. (2013). ¿Cuánto saben de matemática los docentes que la enseñan y cómo se relaciona ese saber con sus prácticas de enseñanza [¿Cuánto saben de matemática los docentes que la enseñan y cómo se relaciona ese saber con sus prácticas de enseñanza]? *Fondo de Investigación y Desarrollo en Educación, Ministerio de Educación de Chile*. https://centroestudios.mineduc.cl/wp-content/uploads/sites/100/2017/07/Informe-FinalF611150-PUC-Beatriz-Rodr%C3%ADguez.pdf

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

Siles, T. (2024). Adidactical situation for a homothetic transformation in the Euclidean plane [Adidactical situation for a homothetic transformation in the Euclidean plane]. *Revista Chilena de Educación Matemática, 16*(3), 89-107.

Stake, R. E. (2007). *Investigación con estudio de casos* [Case study research]. Morata.

Verdugo-Hernández, P., & Caviedes, S. (2024). The work of a prospective high school teacher in pre-professional training in highlighting mathematical knowledge. *EURASIA Journal of Mathematics, Science and Technology Education, 20*(10), Article em2525. https://doi.org/10.29333/ejmste/15473

Yin, R. K. (2018). *Case study research. Design and methods*. SAGE.