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Advances in the characterization of university students' visual thinking through inspection heuristics and representation

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Abstract
Technological advances and the invasion of images impose new educational challenges where it is necessary to use pedagogy of and with images to strengthen the teaching and learning of mathematics and to promote problem-solving strategies. The objective of this article is to present some inspection heuristics and representations used by university students when solving non-routine problems in the context of a mathematical problem-solving course. The methodology was qualitative, with the selection of participants, application of problems and analysis of data for heuristic evidence and representations. The results are part of a broader research that seeks to advance in the characterization of visual thinking in the teaching and learning process of mathematics at the university level. First-year students from three campuses of the Antonio Nariño University in Colombia participated in the qualitative study. This article provides a definition of visual thinking and an analysis of what we call inspection heuristics. Among the inspection heuristics used by students in visual thinking processes, object decomposition, coloring, demarcation with pictograms, comparison/contrast, symbolic demarcation, and others were identified, which are explained in the article. Each heuristic used by students has a purpose in problems-solving.

Keywords: visual thinking, visualization, inspection heuristics, representations, problem-solving, mathematical cognition

INTRODUCTION

Mathematical thinking, its structure, cognitive and developmental processes are of interest to researchers in mathematics education. It is important to understand how visual thinking processes and the visual interpretation of mathematical objects and processes are generated and developed to solve problems. Studies on mathematical visualization are concentrated to a greater extent in some fields such as calculus (Giaquinto, 2007; Guglietti, 2023; Haciomeroglu, 2007; Haciomeroglu & Chicken, 2012; Haciomeroglu et al., 2010; Hitt & Dufour, 2021; Rodríguez-Nieto et al., 2024) and geometry (Gutiérrez, 1996; Hershkowitz, 1989; Hershkowitz et al., 1992; Lowrie et al., 2020). Recent studies assess the incidence and power of visual thinking in other contexts such as mathematical competitions (Mora, 2024) and for problem solving (Falk de Losada & Taylor, 2022; Lowrie, 2020).

In the literature we find the indiscriminate use of terms such as visualization, visual thinking, visual imagery, diagrammatic reasoning or spatial reasoning (Clements, 1981, 1982; Fernández, 2013; Gutiérrez, 2018), which is why in this article we present a definition of visual thinking and some of its distinctive features. The emergence of the term "visual thinking" is in a different field than mathematics or mathematics education: art (Arnheim, 1986). For Arnheim (1979, 1986) visual thinking is associated with perception and his contributions have been fundamental for the study and understanding of this construct from an epistemological perspective proposed by Giaquinto (2007).

Sholihah and Maryono (2020) state that visual thinking plays an essential role in problem solving at all

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Contribution to literature

- We provide a definition of visual thinking in the context of Mathematics Education.
- The identification of some heuristics that we call inspection heuristics and the analysis of some of the representations that university students use to solve non-routine problems.
- We establish the analysis of two distinctive features to advance in the characterization of visual thinking in university mathematics.

Table 1. Definitions or approach to definitions on visualization and visual thinking

Reference author	Definitions or enclosure to definitions on visualization and visual thinking
Krutetskii (1976) cited by Presmeg (2020)	He conceives logical verbal thought as opposed to pictorial visual thought. In this sense, he also conceives two types of thinkers when the two types of thought are in harmony: abstract-harmonic and pictorial-harmonic.
Gutiérrez (1996)	Visualization in mathematics is the type of reasoning activity based on the use of visual or spatial elements, whether mental or physical, carried out to solve problems or prove properties. Visualization is made up of four main elements: mental images, external representations, visualization processes and visualization skills.
Giaquinto (2007)	Visualization is related to perception, which allows recognition, that is, perceptual concept.
Presmeg (1986, 2006, 2020)	He recognized visuality in mathematics as the degree to which a person uses visual methods when solving problems. Visual methods refer to the use of visual images, with or without diagrams. He defined a visual image as a mental sign used to represent visual or spatial information and directly relates internal and external visual images.
Zimmerman and Cunningham (1991)	They conceive it as a process of transferring objects, concepts, phenomena or processes and their representations to another type of representation. This other type of representation can also be visual.
Arcavi (1999)	It is the ability, process and product of creating, interpreting, using and reflecting on drawings, images, diagrams, in our minds, on paper or with technological tools, with the purpose of representing and communicating information, thinking and developing previously unknown ideas and advancing understanding.
McKim (1980)	Visual thinking is carried out through images obtained through sight, imagination or drawing.
Urchegui (2015)	Visual thinking is defined from a pedagogical perspective as the set of cognitive processes that subjects carry out around visual information. Through visual thinking we can interpret reality to act.

educational levels in the teaching and learning of mathematics, but there is concern that many students do not understand how to visualize and represent problems graphically or geometrically or how to solve algebra, trigonometry, and calculus problems. Schoenherr and Schukajlow (2024) investigate different types of external visualizations, such as visualizations with physical similarity (pictorial to abstract) and three types of visualizations with structural similarity of length, area and relational.

The term "visualization" is often used to refer to the use of physical or mental images for the understanding, manipulation and presentation of mathematical concepts, objects and processes. **Table 1** presents some definitions of visualization and visual thinking found in the consulted literature.

The use of visual thinking is indisputably associated with the use of representations. These constitute the fundamental basis for understanding the notions of visualization and visual thinking, as well as for assessing the potential of the latter in mathematics education and in solving challenging problems. Dreyfus (1991) states that "…mental representations are created in the mind on the basis of concrete representation systems" (p. 31). In addition, they can be symbolic, implying the close and inseparable relationship of signs and their meanings. Elsayed and Al-Najrani (2021) evaluated the effectiveness of augmented reality technology in enhancing visual thinking in mathematics and increasing academic motivation among high school students in Saudi Arabia. In their findings, they suggested incorporating augmented reality into the teaching of mathematics at various educational levels. They also recommended its widespread use in mathematics, with a special emphasis on the area of geometry.

Arnheim (1986) places representations at different levels of abstraction from the perspective of art, and from the perspective of research in mathematics education, they are conceived as commanders of thought and drivers of ideas (Reyes-Santander, 2012). They are important and necessary for mathematical activity (Duval, 2006b) and can be considered as artifacts that make it possible to recognize what students know or are thinking, through the processes of encoding or decoding (Lowrie, 2020).

There is a broad classification of types of representations in literature (**Table 2**), and this possibly leads to them being given a predominant role in the mathematics education community. Representations are

Table 2. Types of representations identified in the consulted literature

Reference author	Type of representation	Description
Lowrie (2020)	Internal representations	Based on images and are what Dreyfus (1991) refers to as particular to each person.
	External representations	Based on diagrams and analysis, they allow the communication of ideas.
	Concrete, visual and schematic representations	They are primitive representations for understanding, which allow finding heuristics to solve problems.
	Analytical representations	Representations based on mathematical analysis and formulas. They promote flexibility in mathematical activity.
Duval (2017)	Representations according to their origin	According to the system that produces and limits them. Representations are implicit in them due to their content.
	Semiotic representations	They are phrases, equations or figurative units that have formation and transformation rules.
	Non-semiotic representations	They are produced automatically in the mind or by means of an instrument.
Presmeg (2006)	Internal representations (mental representations)/ External representations	They are correlated. The so-called visual imagery coming from external representations is the internal representation. Internal representations are what Reyes-Santander (2012) calls thought commanders.
Rojas (2014)	Discursive representations	They use natural or formal language.
	Non-discursive representations	Using geometric figures or Cartesian graphs.
Bruner (1966)	Enactive Representation	Representation made with the movement of the body.
	Iconic Representation	Representation of diagrams, figures, icons, as proposed by Giaquinto (2007), granting a facilitating role.
	Symbolic Representation	Representation through symbols. It is a more evolutionary type of representation.
Smith and Kosslyn (2008)	Surface representation	They are quasi-pictorial and are present in active memory.
	Deep representation	They are representations that are stored in long-term memory.
	Representations of specific modality	They use perceptual systems. These include propositional representations, which means they use language and image representations.
Arnheim (1986)	Representation in replica	These are representations that in mathematics education we can identify as prototypical, which have a high heuristic value, but their careless use can generate obstacles in the understanding of mathematical objects and the generation of new ideas. They are cognitive products of the lowest order.
Gutiérrez (1996)	Mental image	Cognitive representation of a mathematical concept or property supported by visual or spatial elements.
	External representations	Any type of representation (verbal or graphic) that serves to create or transform mental images.
Goldin (1992, 1998, 2020)	External	They are visible or tangible productions and external to the subject that produces them.
	Internal	It refers to constructs, concepts or mental configurations of the subject that produces them.
	Conventional	They are based on assumptions or conventions shared by the academic community.
Hitt and Quiroz (2019), Hitt and Dufour (2021)	Representations that are institutionalized by their nature	They are part of a system of signs, governed by certain rules.
	Representations that are not institutionalized by their nature	Less formal (spontaneous) representations. They appear in the process of solving situational problems and are modified mainly during interactions between students or with teachers.

fundamental in the study and understanding of the use of visual thinking in Mathematics Education, since they constitute the main form of communication of ideas for problem solving.

Some of these types of representations were identified in the responses provided by students when they inspected the problems and reported their results. The types of representations that students use when solving mathematical problems can influence their success (Lowrie, 2020), which is why it is important that problem solving is frequently used in the teaching process to help make the processes of transforming representations more flexible. Some researchers in mathematics education recognize different processes

Table 3. Processes with representations, visualization and visual thinking

Reference author	Type of representation	Description
Lowrie (2020)	Encode/Decode	Decoding is used in the interpretation of representations. Encoding, on the other hand, involves the construction of one's own representations.
Bishop (1980)	Interpreting figurative information (IFI)	It involves knowledge of vocabulary and visual conventions.
	Visual processing (VP)	It involves image transformation, manipulation of visual images, and translation of abstract relationships.
Gutiérrez (1991, 1996)	Visual processes	They are based primarily on Bishop's proposal (1980). They are refined as the student puts visual skills into practice.
Duval (2006a)	Treatment	Transformations in the same representation.
	Conversion	Transformations from one representation to another.

that can be carried out using representations, visualization, and visual thinking (**Table 3**).

The different researchers who propose the processes described in (**Table 3**) refer that developing visual processing capabilities in students is not a minor issue. In addition, they insist on the need to work on these processes in the teaching and learning of mathematics, since it favors the understanding and resolution of problems. Visual thinking in the solution of non-routine problems transcends the use of graphical representation in a utilitarian or auxiliary way; it constitutes itself as a way of thinking that can inspire creative and inspiring solutions as conceived by Arcavi (1999) and Fernández (2013).

In the study we focus specifically on the use of visual thinking in problem-solving in the university environment. Reference is made to visual thinking and not to visualization because we consider the latter to be a distinctive feature of the former. This idea is expanded in the theoretical framework. In the article we identify some heuristics that we call inspection heuristics, and we analyze some of the representations that students use.

Representations are an important distinguishing feature of visual thinking, as they can provide clues about the ways in which students analyze, manipulate, and generate ideas to solve problems. Although there is a wide variety of types of representations in the literature, what is important about them is how they are used and articulated in the solution of non-routine mathematical situations.

CONCEPTUAL FRAMEWORK

The theoretical foundation that guides this study focuses mainly on presenting how we conceive visual thinking, visualization, representations and the relationship of all of them with the solution of non-routine problems in mathematics.

Theoretical Contributions on Visual Thinking

In this study we propose that in mathematics education we use visual thinking as a broader construct that transcends the areas of mathematics and even involves visualization. **Table 1** lists some definitions of visualization that include aspects that are fundamental in visual thinking, but that do not specify in their scope what we conceive as visual thinking in this study.

Arcavi's (1999) definition of visualization is the one that comes closest to what we consider to be visual thinking. In it, it is conceived as a skill; that is, it can be assumed from the start that it is possible to develop it. He also conceives it as a process (Dreyfus, 1991), as a verb (Fernández, 2013), as an active and dynamic construct, and finally as a product, that is, as a result or delivery (communication).

In Sánchez (2024, p. 57) a definition of visual thinking is presented as the:

> *… cognitive process that involves particular and repetitive mental acts associated with the representations of what students imagine, see, perceive and know, together with the structural changes of mathematical objects and concepts that are generated for problem solving.*

This way of understanding visual thinking allows the relationship with the mental act of solving problems that, under repetitive circumstances, can be associated with a way of thinking (Harel 2008, 2009). It also considers other topics frequently presented lightly in the definitions in **Table 1** such as imagination, so closely related to creativity that it is implicitly required for solving mathematical problems, the perceptions and prior knowledge of students.

All the researchers consulted agree that representations play a very important role in mathematical thinking and the solution of non-routine problems. The definition includes structural changes that refer to the effectiveness and refinement of the student's cognitive structure to the extent that he or she makes use of visual thinking and not just visualization. Collectively, visualization is understood as the act of "seeing", of approaching a representation in a partial way and generating a partial interpretation if other distinctive features of visual thinking do not converge. When referring to distinctive features, they are the attributes or qualities that characterize visual thinking and that currently continue to be a topic of interest for researchers in mathematics education.

In this context, visualization must always be accompanied by other processes such as the refinement of intuitions and perceptions, as well as the strengthening of the student's mathematical scaffolding or structure to solve problems.

Theoretical Contributions to Representations

As can be seen in **Table 2**, a wide variety of types of representations and their processes can be found in the literature. The discussion on these aspects has been open for decades and continues to arouse the interest of researchers as new possibilities open in the field of technology and artificial intelligence.

What is indisputable is that mathematical ideas must have some form of representation for their manipulation and communication. It is also indisputable that representations are fundamental in the resolution of mathematical problems and can become drivers of ideas (Arcavi, 1999; Reyes-Santander, 2012) and heuristics for problem solving. They also constitute a stage for the ingenious communication of ideas that can help in understanding mathematics. From the types of representations listed in **Table 2** for this study, it was initially defined to identify institutionalized replicas, surface, symbolic or iconic representations. We identified that some types of representations proposed by researchers may be talking about a similar type of representation, for example, conventional representations (Goldin, 2020) and institutionalized ones (Hitt & Dufour, 2021).

Some of these types of representations are identified in the responses provided by students when they explored the proposed problems and communicated their results.

Theoretical Contributions on Problem Solving and Visual Thinking

Problem-solving is a key component of reasoning (Arcavi, 1999), a way of thinking (Harel, 2008) that requires the use of some kind of representation. The contributions of Polya (1945, 1973) and Schoenfeld (1985, 2022) with their methods, heuristics and determinants for problem solving constitute an important source of reflection and analysis in the broader study of which the content of what is presented in this article is part.

This article reviews what have been called inspection heuristics to refer to the paths, routes, tactical decisions - in the sense of (Schoenfeld, 1982) - or rules to advance in problem solving (Polya, 1945), during initial exploration.

In this article, inspection heuristics are the ways in which students' approach and deal with information about a problem. According to Schoenfeld (1982, 1985), various heuristics can be established in the exploration phase. However, successful problem solving requires what he has called self-regulation. We believe that the use of visual thinking in the stage of approaching and understanding a problem, which we have called inspection and is part of the exploration stage, can help to trace the route to find a solution.

The following are inspection heuristics using visual thinking: image decomposition or visual decomposition, complementation, coloring, numbering, symbolic demarcation. These heuristics facilitate visual inspection and can help direct the solution to a problem. Inspection heuristics and representations are what we call in our research distinctive features of visual thinking in mathematics. In the study where we intend to advance in the characterization of visual thinking in mathematics, we identified at least six distinctive features. In this article we will focus on only two of them.

METHODOLOGY

The information presented in this article corresponds to a broader study that was carried out with the aim of advancing the characterization of visual thinking in the context of a mathematical problem-solving course. A literature review was carried out that focused on identifying, first, the discussions and advances that have taken place in events such as the International Congress on Mathematics Education (ICME) and the Conference of the International Group for the Psychology of Mathematics Education (PME). The proceedings of the ICME since 1972 and of the PME since 1978 were reviewed.

To identify quality publications, databases were consulted with search equations that combined key words such as visual thinking or visualization and education; visual thinking or visualization and mathematical problem solving. In the second stage of the research, the research focused on combinatorial representations and mathematical problem solving.

This qualitative research, of which only some results are presented in this article in relation to the inspection heuristics and the use of representations by students, was developed in three cyclical phases. The first phase involved approaching experts, the second was an exploratory phase, and the third was a delimitation and progress in the characterization of visual thinking. In the exploration phase, an activity with three non-routine problems was structured and applied considering the contents established in the curriculum, to validate the effectiveness of the activity. The exploratory activity was reviewed by an in-service mathematics teacher and researcher from the University of Playa Ancha (Chile).

Participants and Context

Sixteen university students (eight women and eight men) from the first year of biomedical and electronic engineering at the Neiva and Popayán campuses of the Antonio Nariño University of Colombia participated in this research, with the aim of advancing the characterization of visual thinking. It should be noted

Figure 1. Metal profiles (Source: Authors' own elaboration)

that these students (5 from Neiva and 11 from Popayán) were enrolled in the Mathematical Problem-Solving course offered by the University for all academic programs to develop quantitative reasoning, critical thinking, research skills and promote mathematical understanding and the transition between high school and higher education. This study began in 2021, but the implementation of the activities took place in 2023 and 2024.

The authors of this research informed the participants about the development of this project for educational and non-economic purposes, which allowed for reliability and openness towards participation in the development of the activities.

Data Collection: The Mathematical Problem-Solving Course

The Antonio Nariño University in Colombia offers several Engineering Programs that include in their curriculum, as a mandatory subject, the course on solving mathematical problems. Based on the course content, the course seeks to promote the understanding of mathematics, and based on the activities, to develop mathematical thinking. This course is not conceived as a precalculus but rather seeks an adequate transition between school mathematics and that of the tertiary level. Given its purpose, the course content is varied: number theory and combinatorics, algebra, geometry, linear algebra, among others. The pedagogical strategy of the course is the solution to mathematical problems.

Six learning guides were structured with the following structure:

1. Title of the guide,
2. Objective,
3. Methodological
4. Suggestion
5. Materials to be used and
6. Proposed problems.

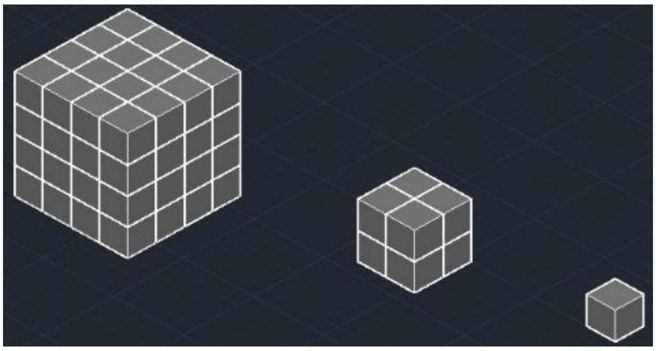

Figure 2. Cubes (Source: Authors' own elaboration)

With the exploratory activity it was confirmed that the structure of the guide was adequate for the time established for the application of each activity and the purpose of the development of the guide. The proposed problems were reviewed by mathematics teachers with extensive experience in the mathematics problem solving course of the Neiva Section and research professors in mathematics education from universities in Colombia, Brazil and Chile. In the design of the activities, graphic, verbal or verbal problems accompanied by a graph were considered.

This article presents the results obtained from Activity No. 3, which proposes the following problems:

Problem 1: Balanced transport

In a metal profile workshop, 36 T1 profiles, 24 T2 profiles, 12 T3 profiles and 6 T4 profiles must be transported. They have 2 vehicles for this purpose, but they must balance the weights. Please indicate at least 3 combinations between the types of profiles, to transport all the profiles, loading the same weight in each vehicle? (see **Figure 1**).

Problem 2. Count the cubes

How many cubes of each size can be identified in the following construction? (see **Figure 2**).

Problem 3. Count the paths (see **Figure 3**).

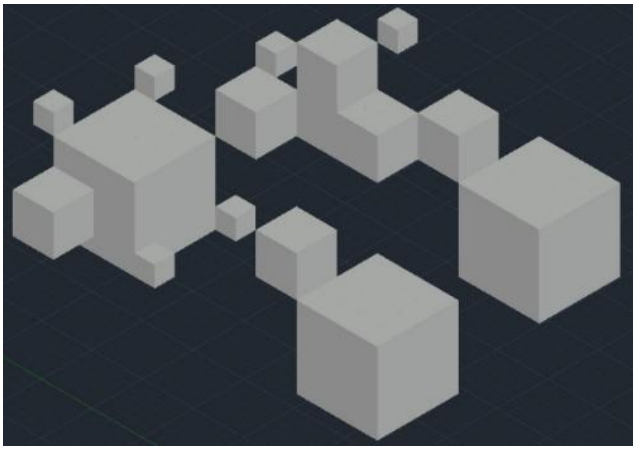

Figure 3. Cube size in construction (Source: Authors' own elaboration)

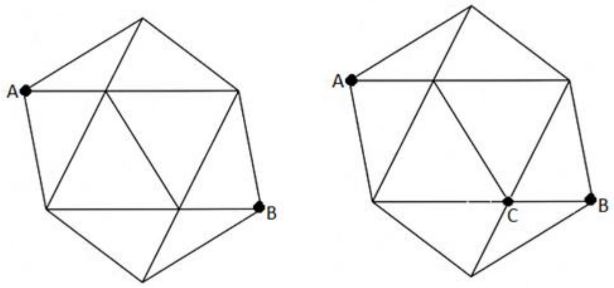

Figure 4. Paths, edges and vertices (Source: Authors' own elaboration)

How many paths are there to get from A to B (see **Figure 4**) walking along the edges? How many paths are there from A to B, without going through C, walking along the edges? How many paths are reduced if the passage is limited by another vertex, walking along the edges?

The sixteen students who participated in the activity developed a guide in the class for the mathematical problem-solving course at each of the university campuses. The professor in charge of the course accompanied and guided the development of his class, which were videotaped as well as subsequent conversations with the professors in charge of the courses in Neiva.

Data Analysis

The goal of the study was to advance in the characterization of visual thinking in the context of a mathematical problem-solving course; however, in this article we will focus on the use of visual thinking from the inspection heuristics and representations. In the analysis process, the students' written responses, the conversation with them during the development of the activities and the review of the class videos, as well as the conversation with experts and with the professors of the mathematical problem-solving course were triangulated.

To analyze the written responses, we generated an instrument called a "rubric for the characterization of visual thinking." In the rubric, we related the distinctive features of visual thinking and its attributes. Each activity was analyzed considering the attributes identified in the material submitted by the students. **Table 4** presents the types of representations that were initially considered and the inspection heuristics.

Table 4. Inspection heuristics and representations

Inspection heuristics	Representations
Decomposition	Institucionalizadas
Complementation	In reply
Coloring	Surface
Numbering	Symbolic
Symbolic demarcation	Iconic
Classification of objects	Prototypical
Comparison of objects	Semiotics

During the implementation of the activities and the triangulation of information, the list of heuristics and types of representations was expanded, as will be seen in the results. It is worth mentioning that the heuristics were defined and agreed upon considering the review of the literature on visual thinking and representations, which was validated by expert researchers in the problem solving and visualization course.

RESULTS

The first problem contains a graphical representation, a description of the situation and the problem questions, which involve notions of equalities and inequalities. We see that all sixteen students performed the inspection by classification and comparison of objects (**Figure 5a**). One student used the coloring heuristic (**Figure 5b**).

It is observed that coloring was used with an initial purpose of organizing information and anticipating the resolution strategy used by the student (**Figure 6**).

Students favored the use of symbolic representations, as shown in **Figure 7**. Some of them resorted to the use of formulas or other arithmetic processes. It was identified that the iconic or symbolic representations used by students are important to understand the paths they take in solving a problem-what they are thinking-and what they really know. Some students, for example, worked only with equalities. In the research we found that indeed the representations used from the inspection of the problem can be important artifacts in the sense expressed by Lowrie (2020).

Two students generated a graphical representation (**Figure 8**), which we identified as replication representations.

Regarding the second problem, students performed the inspection mostly by numbering and/or decomposing/subdividing the given objects. One student generated an auxiliary representation, while the remaining fifteen did not generate additional representations (**Figure 9**).

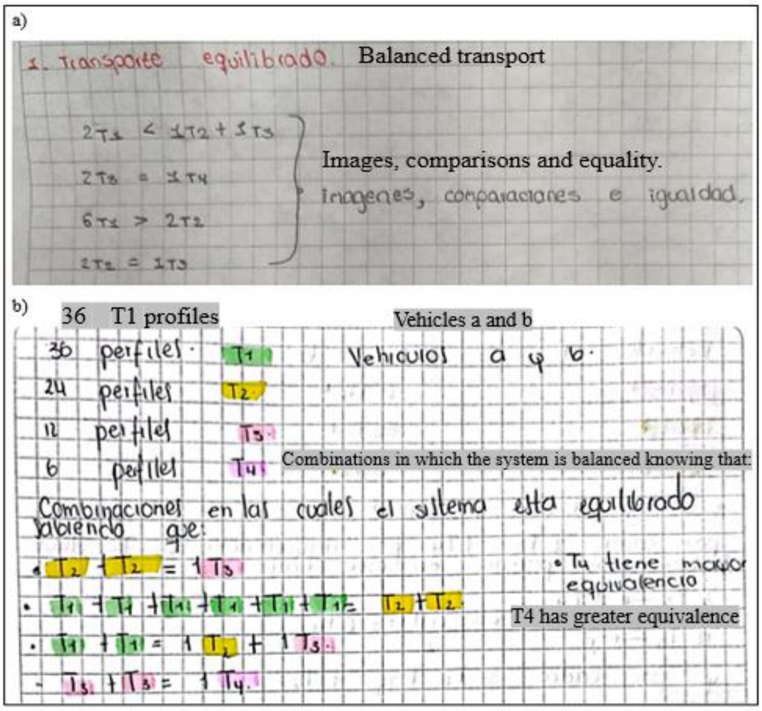

Figure 5. Examples of inspection by a) decomposition and/or comparison and b) coloring, by two students (Source: Authors' own elaboration)

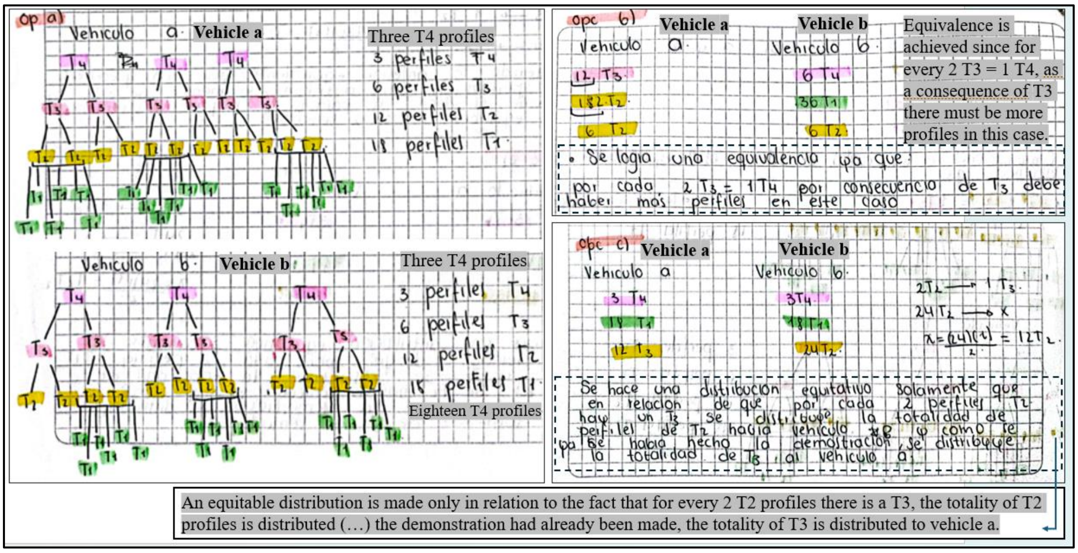

Figure 6. Representations that emerged from a student's inspection-by-coloring heuristic (Source: Authors' own elaboration)

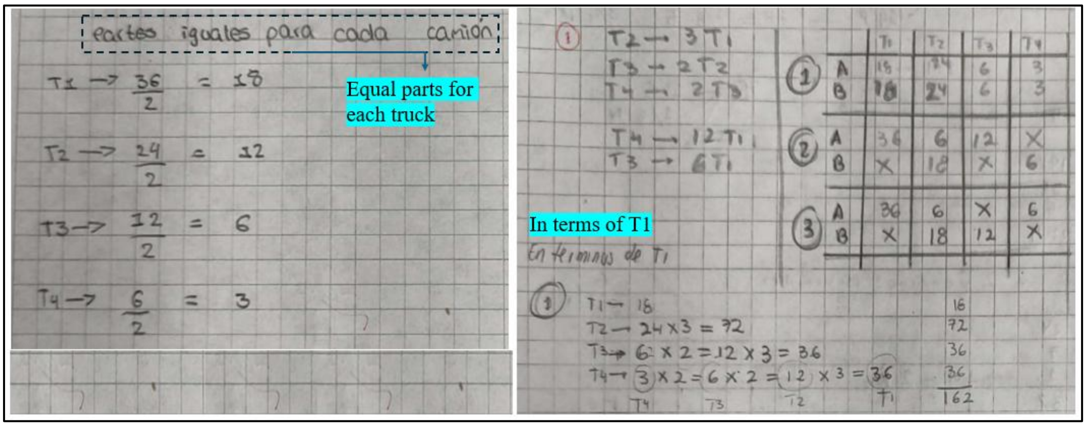

Figure 7. Examples of symbolic representations from two students (Source: Authors' own elaboration)

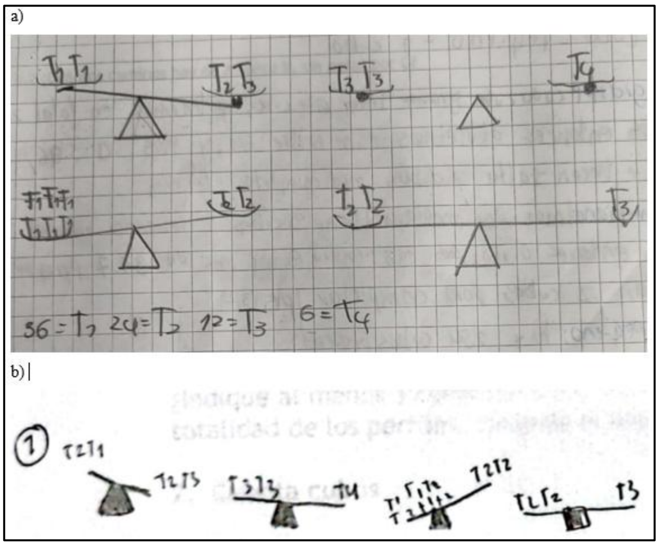

Figure 8. Graphic representations made by three students (Source: Authors' own elaboration)

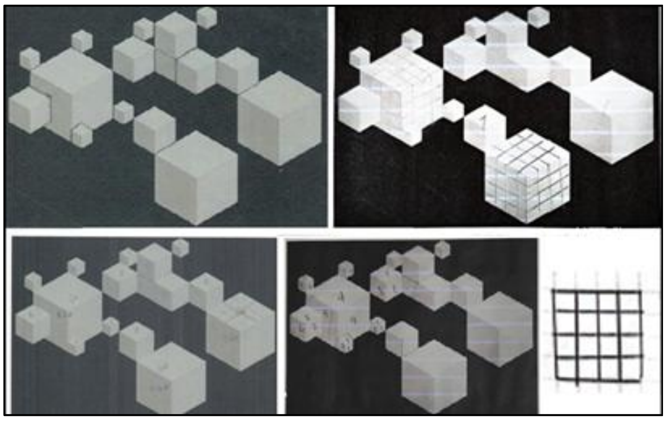

Figure 9. Decomposition/subdivision of objects (Source: Authors' own elaboration)

The students began by counting the objects given and then, using multiplicative operations, they gave their answer to the problem. One student generated the answer to the problem in terms of the total number of cubes for each size, but it is not evident how he advanced the inspection process.

During the activity, the students from Neiva consulted the possibility of not only decomposing into smaller cubes, but also of identifying how many larger cubes could be identified by joining the objects. They were not given this possibility, however, the students from Popayán were, so some students, once they had obtained all the smaller cubes, carried out divisions to achieve the formation of new cubes of each size. Seven students only made symbolic representations (**Figure 10**), six students carried out the inspection by decomposition or subdivision of objects and four also used the numbering of objects.

The third problem of this activity provided rich representations, both graphic and symbolic, in the treatment and generation of results. The problem generated by graphic representation was inspected by the students in various ways: using coloring, numbering and what we call demarcation with pictograms mainly (**Figure 11**).

We found that seven students made auxiliary graphic representations, five of them made replica representations (**Figure 12**). In the heuristic inspection process, representations are drivers of ideas and a source of ways to solve non-routine problems.

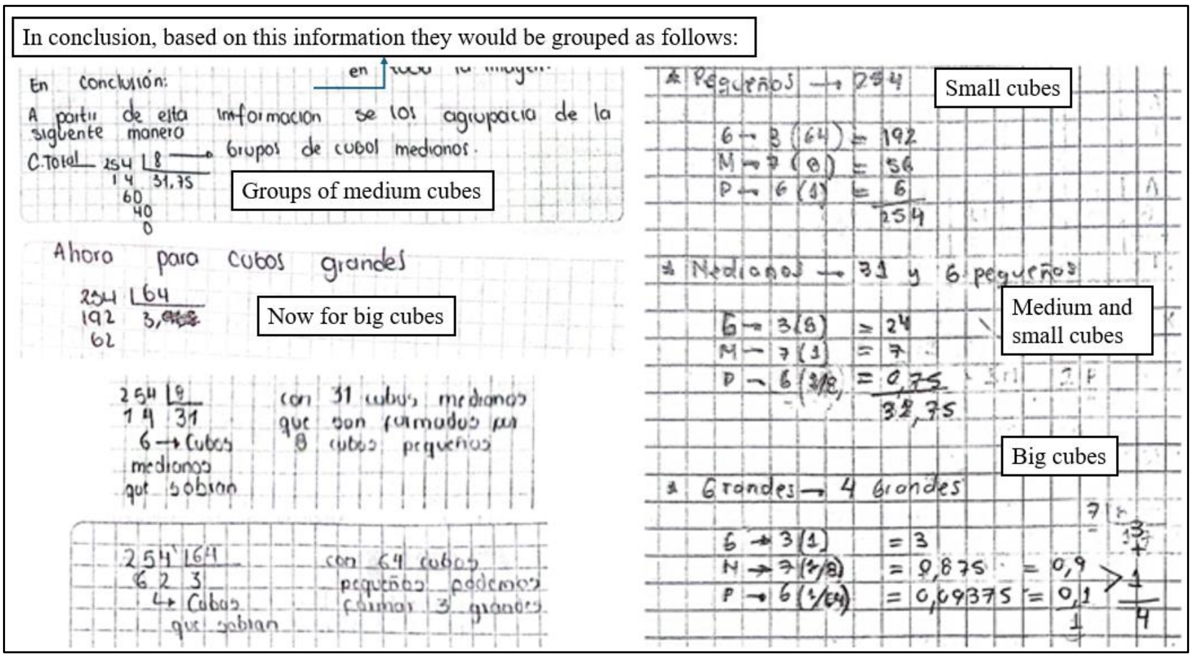

Figure 10. Examples of mathematical operations performed by students to solve problem 2 (Source: Authors' own elaboration)

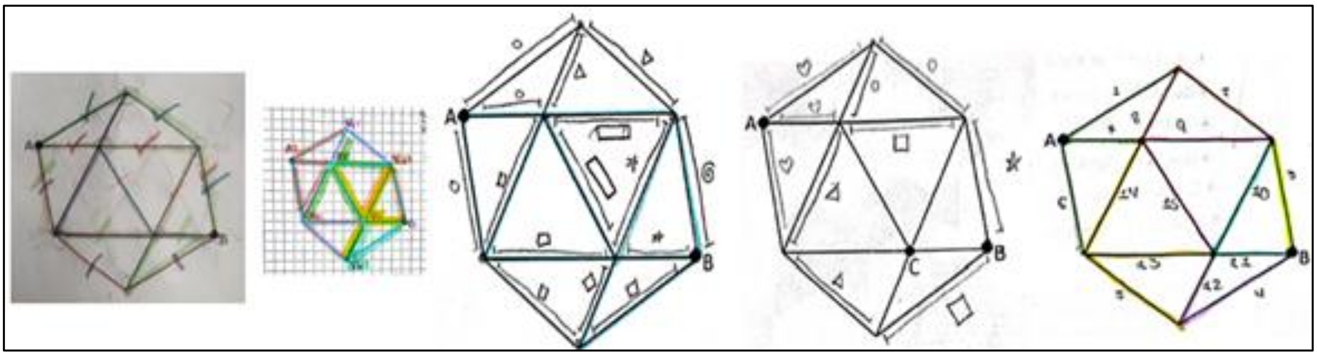

Figure 11. Examples of inspection processes for problem 3, carried out by students (Source: Authors' own elaboration)

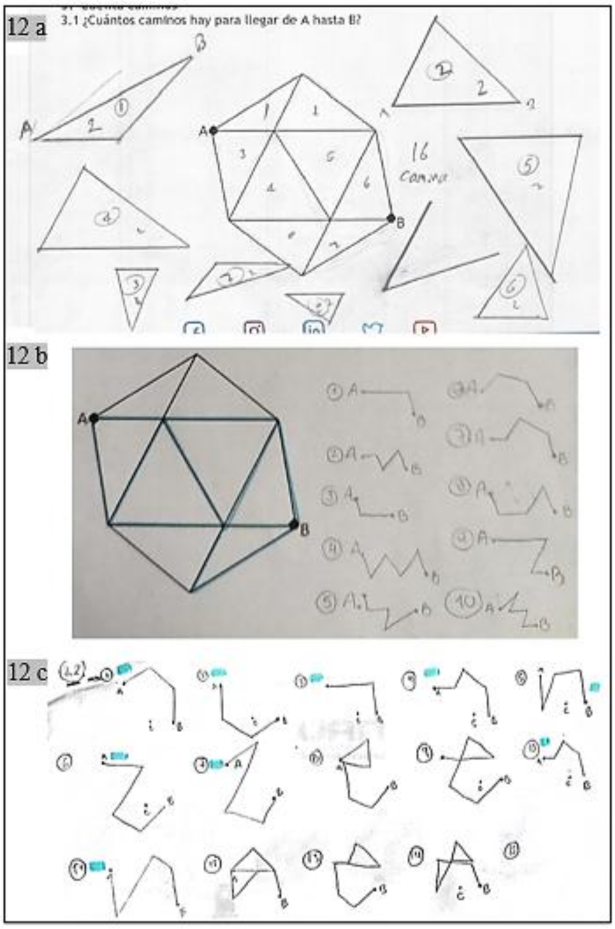

Figure 12. Examples of student-generated replication representations for Problem 3 (Source: Authors' own elaboration)

Two students made three diagrams (**Figure 13**) to organize information. One of these auxiliary representations was also accompanied by coloring. One of the students explains that the purpose of making this diagram is to avoid repeating lines.

Three students used pictogram marking as a form of inspection to arrive at a solution to the problem (**Figure 14**). Pictogram markings serve an organizational purpose to establish paths. Making use of visual information to interpret and lead to action is part of the construction of visual thinking (Urchegui, 2015) and is conceived in this research with a fundamental purpose of the same.

Eight students used color inspection (**Figure 15**) to facilitate the identification of the routes followed on each path.

Inspection heuristics and representations are distinctive features of students' visual mathematical thinking, as they allow them to identify ways of thinking when solving non-routine problems. We found in the application of the activity that students use the decomposition of objects to identify geometric shapes of other sizes or to facilitate the counting of objects (**Figure 16**).

The student who performed the decomposition of problem 3 into triangles (**Figure 16**), used this strategy as a second option, after insisting that he could not solve the problem because he did not have a formula for it. His strategy was to decompose the problem into smaller parts, since he "did not find a formula" to solve the problem. It is important to identify how students give relevance or importance to the constituent parts of a representation, given that as Merleau-Ponty (1993) refers, what is perceived is always part of a broader field, which is why the recommendations of problem solvers should be followed, in terms of dividing problems into simpler parts or looking for similar ones that have been previously solved (Schoenfeld, 1985).

Coloring and marking with pictograms are used to differentiate objects, facilitate counting, identify directions of movement or establish order. **Figure 14** shows markings with pictograms (squares, hearts, stars) and **Figure 17** shows the use that the student gave to these markings.

The intentionality of the coloring is also evident in what one of the students did to solve problems 1 and 3 (**Figure 18**).

The student used coloring in the inspection process to organize auxiliary representations. We identified these representations as external (Gutiérrez, 1996), non-institutionalized (Hitt & Dufour, 2021), or idiosyncratic (Goldin, 1992, 1998, 2020). Numbering and symbolic demarcation were used as inspection heuristics mainly in problem three to establish order issues (see **Figure 13**, **Figure 15** and **Figure 19**).

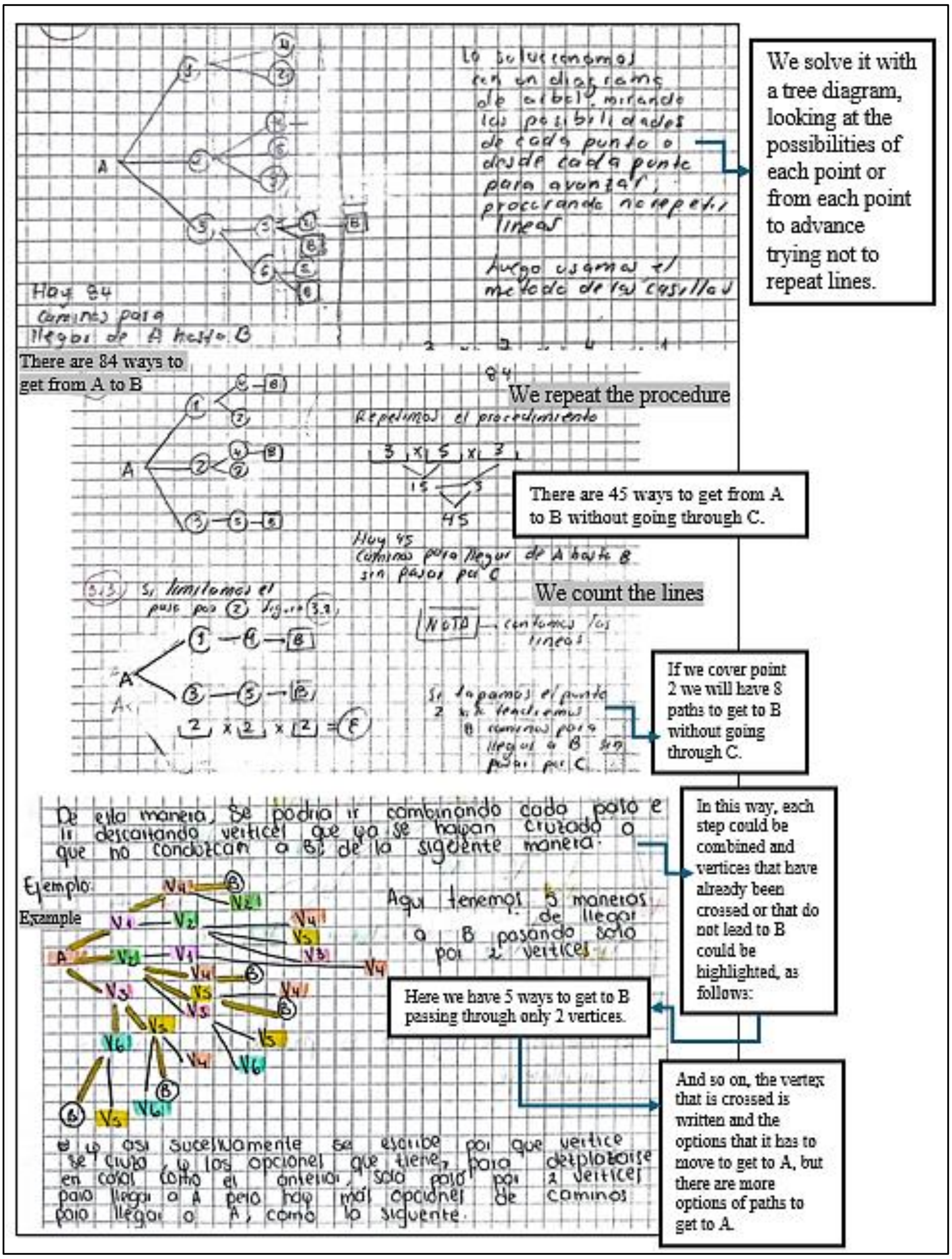

Figure 13. Examples of auxiliary representations (tree diagrams), to solve problem 3 (Source: Authors' own elaboration)

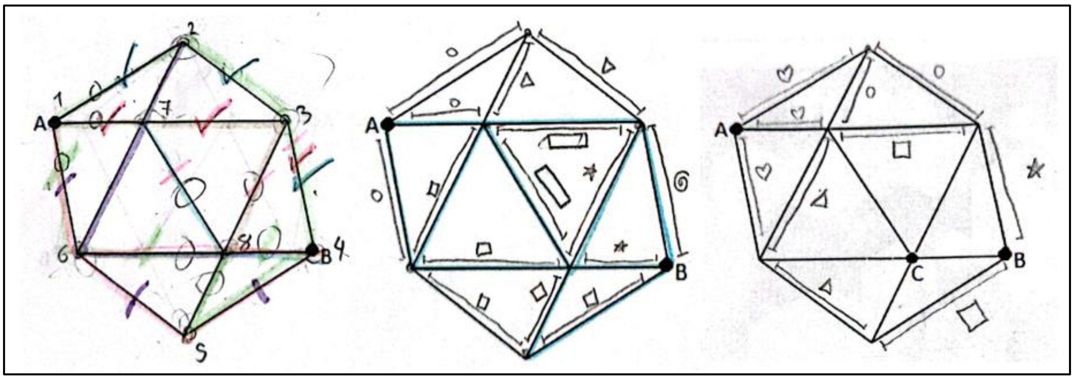

Figure 14. Examples of student-generated representations for problem 3 (Source: Authors' own elaboration)

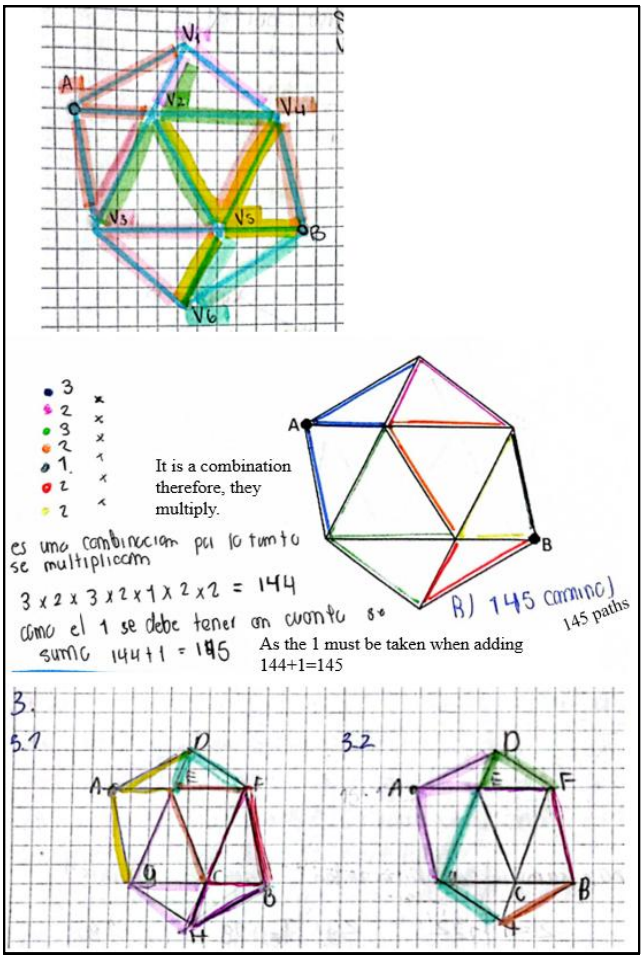

Figure 15. Examples of color inspections for problem 3 (Source: Authors' own elaboration)

Regarding representations, some institutionalized ones are evident in the development of problems one (**Figure 20**) and two (**Figure 21**). Examples of this are the symbolic representations of arithmetic operations to find the number of cubes of different sizes, the representations of equalities or inequalities in problem one and the decomposition into prime numbers, in the same problem. We find symbolic representations that are used when students find a connection with some mathematical procedure (products to represent successive sums, formulas to determine quantities) or concepts (numerical sets, equalities, inequalities).

DISCUSSION

Inspection heuristics and representations are important and powerful hallmarks of visual thinking for solving non-routine problems in mathematics. In this activity we found that replica representations were most evident in problem three. These representations helped to find paths, facilitate counting, and identify diagram elements; however, they are also representations that can limit the generalized solution to a problem, as occurred in other activities applied during the research.

Although visual thinking is not yet considered a tool for accessing knowledge (Urchegui, 2015), we have shown that the use of multiple representations in the analysis of a problem allows students to make their ideas more flexible, relate previous concepts and find ingenious solutions.

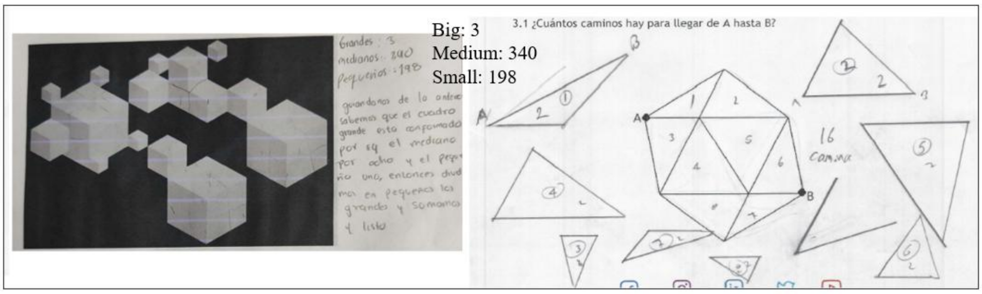

Figure 16. Examples of decomposing objects for counting. Examples from two students are presented (Source: Authors' own elaboration)

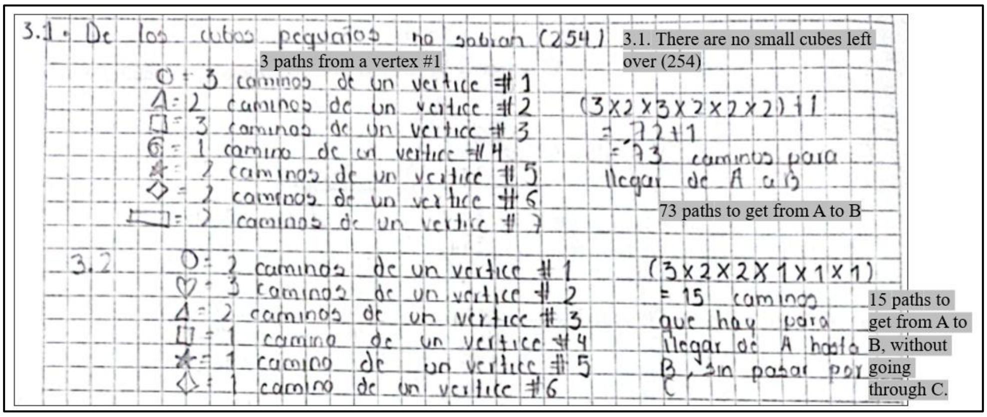

Figure 17. Examples of how the pictogram demarcation was used by one of the students to answer problem 3 (Source: Authors' own elaboration)

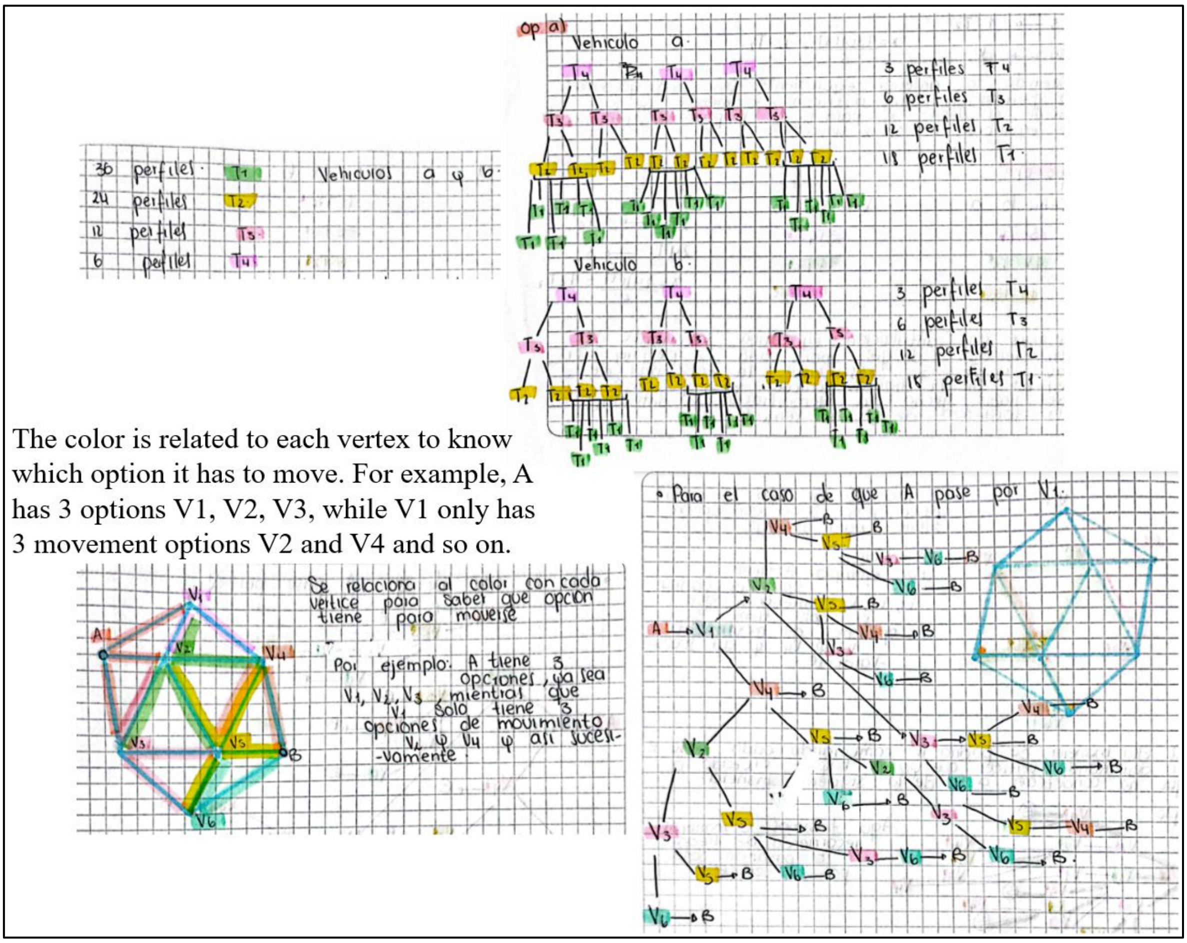

Figure 18. Using coloring as an inspection heuristic to build auxiliary representations (Source: Authors' own elaboration)

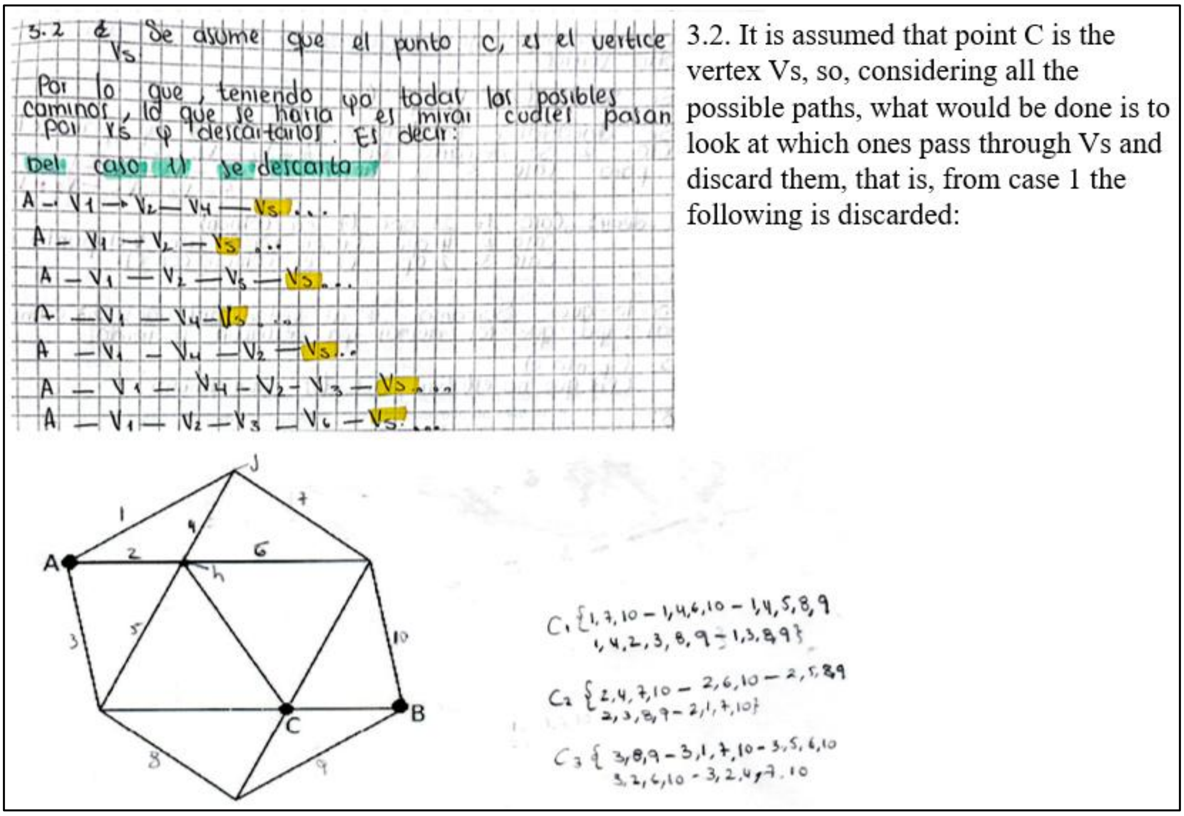

Figure 19. Using coloring as an inspection heuristic to build auxiliary representations (Source: Authors' own elaboration)

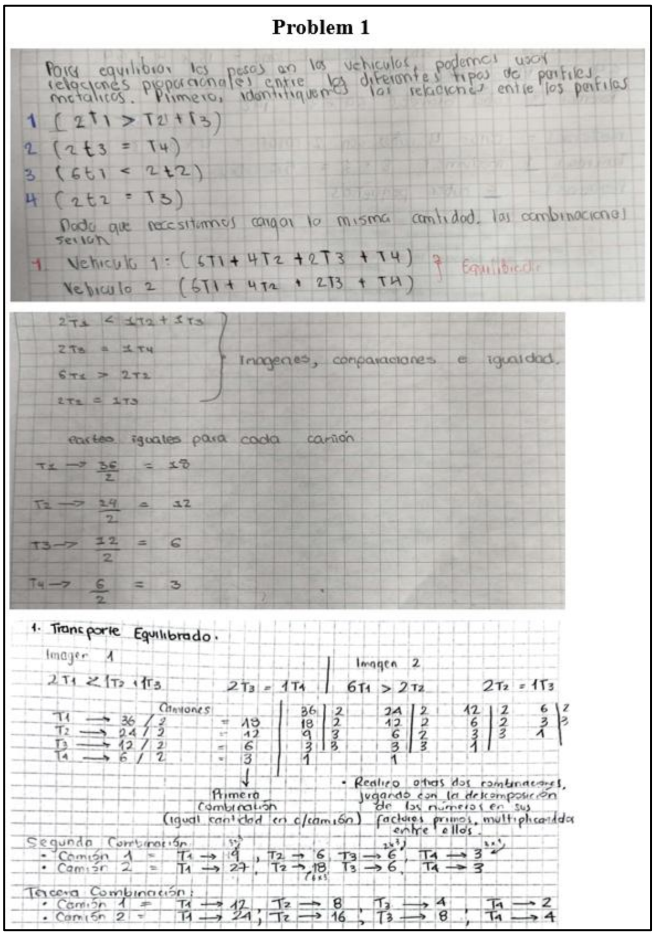

Figure 20. Examples of institutionalized representations from problem 1 (Source: Authors' own elaboration)

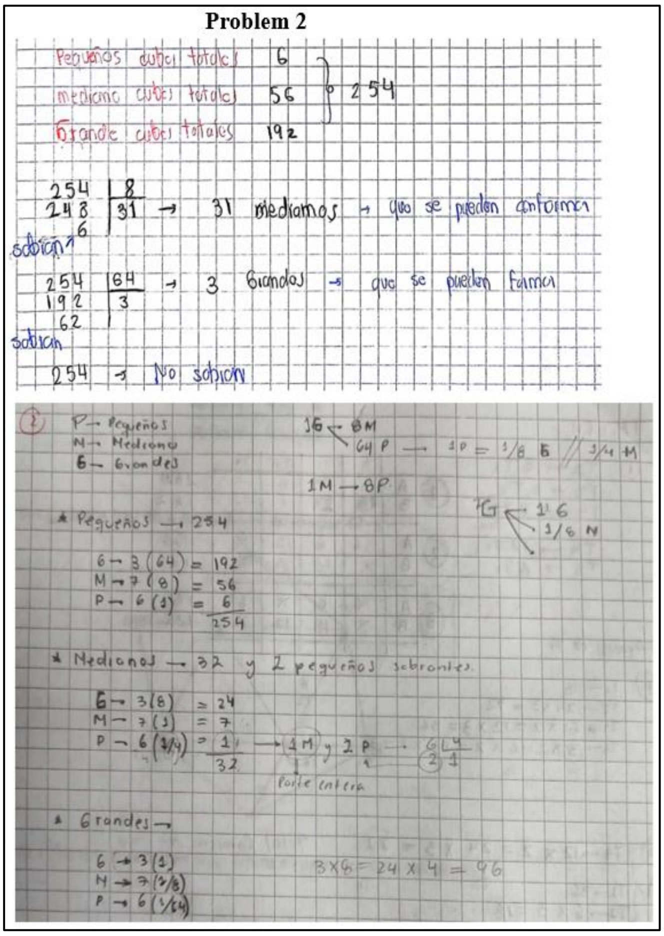

Figure 21. Examples of institutionalized representations from problem 2 (Source: Authors' own elaboration)

Visual thinking is important in solving non-routine problems and, as identified by Wiseman and Mudaly (2023) based on the approaches of Sfard (2008), students have internalization processes, where they individually engage in dialogues and find ways to visualize and represent the information of a problem. Inspection heuristics can change as the view is refined, as occurred with the student from Popayán who used coloring, numbering, and symbolic demarcation to make the tree diagrams in problems two and three of the guide. This student frequently uses this form of inspection and representation of the problem, as it allows her to organize the information.

Symbolic representations are more frequent in problems that evoke the development of institutionalized strategies, such as working with equalities in problem one or additive and multiplicative operations in problem two. Non-institutionalized representations, on the other hand, are circumstantial as Hitt and Dufour (2021) have pointed out, but they require creativity and ingenuity. Inspection heuristics and representations made by students to solve a non-routine problem require, in some way, an intellectual effort, resorting to prior knowledge or to ingenious ways of numbering, coloring, dividing or complementing representations. We did not see the use of complementation in the activity developed, however, in the broader study this strategy was mainly used with geometric problems.

It was identified that students' particular and repetitive representations can indicate ways of thinking when solving non-routine problems. The symbolic demarcation used by students, which sometimes seems meaningless to teachers, can evoke relationships with prior knowledge or be used as a strategy for organizing ideas.

CONCLUSIONS

This research shows a special analysis of visual thinking and constitutes a precedent for future studies because it emphasizes some inspection heuristics and representations used by students when working with non-routine problems where visual thinking is used. In addition, it shows the way to advance in the characterization of visual thinking in Mathematics Education, particularly at the university level, since we emphasize fundamental features of such thinking.

The first ideas that students have when analyzing a non-routine problem constitute paths that may involve different representations, which may change at different stages of the solution. We have shown that the initial representation usually has the purpose of organizing information, while the subsequent representations can be used to generalize, formalize or present solutions or beautiful ideas.

We agree with Fernández (2013) and other researchers in the broad research agenda in Mathematics Education, on the need to continue clarifying the terms visualization and visual thinking and representation processes (Cantillo-Rudas et al., 2024; Font et al., 2024; Galindo-Illanes et al., 2025; Ledezma et al., 2024; Rodríguez-Nieto et al., 2023; Rodríguez-Nieto et al., 2024), as an opportunity to advance in the identification of distinctive features that allow us to find ways of thinking visually in mathematics. Problem solving as a way of thinking requires visual thinking and multiple cognitive tools for the search for creative solutions. This implies that teachers must pay more attention to the processes that students develop in the search for these solutions, since sometimes bottlenecks or difficulties arise because they use a single form of representation.

In the research we have found other distinctive features of visual thinking, such as intuition, perceptions, creativity and communication. The union of the distinctive features has allowed us to structure a model scheme for characterizing visual thinking, where representations are fundamental and articulating all these distinctive features. We suggest that future research should delve into the limitations generated by the frequent use of institutionalized representations in the teaching process at all levels of mathematics education. We believe that the study of visual thinking has a long way to go, especially with the challenges related to artificial intelligence and the saturation of the immediate, the visual and the iconic, in mathematics education. A limitation of this work could be the type of tasks because they are not the only rich tasks to explore visual thinking, contextualized tasks can also be designed and proposed where ethnomathematical, didactic and globalization aspects are considered (Rodríguez-Nieto & Alsina, 2022).

It is also suggested to deepen on the power of visual thinking in other areas of mathematics at different educational (Nápoles & Rojas, 2020) and university levels such as differential equations and even probability and statistics that require the competent use of visual aspects and different types of representations and connections between them.

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