

A synopsis of the importance of teaching fractions to children until K-10

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Abstract

This study investigates the pedagogy and evaluation of fractions throughout the educational progression, ranging from early childhood education to the K-10. The article is presented from fundamental fraction notions, with concrete illustrations, through more complex numerical operations. The introduction addresses mathematical and ordinary contexts, while the primary emphasis is on the efficacy of using visual aids and real-world illustrations to enhance comprehension. The research furthermore examines a range of evaluation techniques, including formative assessments and technology-enhanced evaluations, to obtain a comprehensive understanding of students' skills. The highlighted instructional routes include a range of approaches, focusing on personalized learning and prompt feedback. The use of sophisticated digital technologies, such as virtual and augmented reality, presents a potentially fruitful path for enhancing the effectiveness of fraction instruction. Although conventional approaches have established a solid foundation, integrating additional digital tools can improve students' learning experience.

Keywords: fraction education, mathematical development, teaching progression, assessment strategies

INTRODUCTION

Main Goals of the Research Paper

Comprehending the fundamental principles is something crucial for students' future in mathematics. According to the National Mathematics Advisory Panel (2008), a solid understanding of foundational mathematical concepts, like fractions, is needed for advanced mathematical reasoning and problem-solving. Having in mind this, and the importance of fractions in several areas of mathematics and their wide-ranging applications, we want to investigate how children in grades K-10 understand fractions as linkages between parts and wholes. Our emphasis will be on identifying the specific obstacles they face. According to the findings of Bailey et al. (2014), early knowledge of fractions may be used to predict children's performance with more complex mathematical concepts and overall academic accomplishment. So, understanding this more deeply is even more important and urgent. Also, along with the study, we want to present possible solutions to those

challenges by analysing how students could deal with this concept inside the classroom and what types of tools the teacher has to teach this complex concept. Ultimately, we want to improve the understanding of fractions and their teaching, improving mathematical education and students' academic achievement.

Mathematical Definition and Importance of Fractions

Bennett and Nelson (2017) define fractions as numerical expressions representing a part or multiple portions of a whole. Their essential is to represent quantities that are not whole numbers, allowing for mathematical precision in conveying and manipulating such quantities. The structure of a fraction consists of a numerator representing the divided parts and a denominator representing the total number of divisions of the whole (**Figure 1**). **Figure 1** illustrates the basic structure of a fraction, showcasing how it represents a portion of a whole. It emphasizes the components of a fraction: the numerator and the denominator.

Contribution to the literature

- This article provides a comprehensive analysis of fraction instruction techniques from early childhood to K-10, and it will fill a gap in the existing literature on the progressive teaching methodologies of fractions in Mathematics Education.
- By integrating advanced digital technologies such as virtual reality (VR) and augmented reality (AR) into fraction instruction, this study highlights innovative strategies that can enhance student engagement and understanding of this very important aspect. Thus, the article offers new perspectives on the use of technology in Mathematics Education.
- The vision of the formative and technology-enhanced assessments for fractions brings new insights into effective evaluation methods and could contribute to the development of stronger and adaptive teaching practices in daily activities.

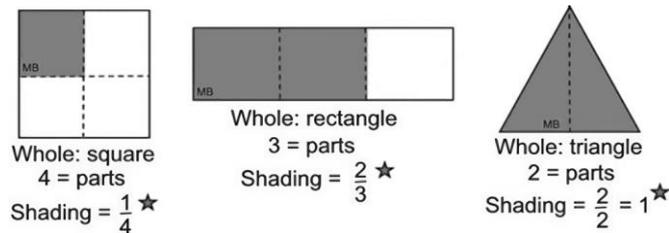


Figure 1. Portion of a whole (Source: <https://mathbitsnotebook.com/JuniorMath/FractionsDecimals/FDfractionbasics.html>)

Fractions provide the capacity to express, manipulate, and reason with non-whole numbers, expanding the numerical system beyond integers (Fischbein et al., 1985). This expansion proves indispensable in modern calculations and problem-solving, where mathematical precision and the ability to manage complex numerical scenarios are essential.

Their significance extends to multiple mathematical disciplines. Fractions are central in algebra, geometry, calculus, probability, and statistics (Thompson & Saldanha, 2003). They occupy the voids between integers on the number line and provide the continuity required for mathematical coherence and the development of mathematical theories and applications.

Applications of Fractions in the Real World

Again, Bennett and Nelson (2017) assert that fractions are a fundamental aspect of ordinary life, transcending their academic significance. They are the foundation of numerous everyday calculations in our lives:

- Cooking and baking: recipes frequently call for fractional quantities, such as $\frac{1}{2}$ cup of sugar or $\frac{3}{4}$ cup of milk. When doubling or halving a recipe, fractions must be multiplied or divided;
- Money: is a practical way to understand fractions. A quarter represents $\frac{1}{4}$ of a dollar, a dime represents $\frac{1}{10}$, and a nickel represents $\frac{1}{20}$ (considering the USA coin extended easily to all the other currencies in the world);

- Time: we frequently use fractions to express time. For instance, half an hour is half of an hour, and fifteen minutes is a quarter of an hour;
- Home improvement: when measuring for new furniture, calculating the area of a room for painting, or cutting timber for a project, fractions are frequently required.
- Sports: fractions are used to compute batting averages in baseball, shooting percentages in basketball, and horse racing probabilities, among several other sports;
- Art and design: when constructing or interpreting visual designs, fractions may describe the proportions of various elements;
- Music: notes in music are frequently described using fractions. Example: a whole note equals $\frac{1}{1}$, a half note equals $\frac{1}{2}$, a quarter note equals $\frac{1}{4}$, etc.;
- Travel: once examining your vehicle's fuel gauge, fractions indicate how much fuel is left in the tank;
- Shopping: discounts, promotions, and comparative purchasing frequently involve fractions. For instance, a quarter-off or fifty-percent-off sale or determining which product is the best value based on size and price; and
- Health and fitness: diet (portion sizes), medicine (dosage instructions), and physical exercise (dividing an hour into fractions for various exercises) all use fractions.

This extensive application of fractions highlights the significance of mastering them, not only for their mathematical significance but also for the practical implications of this mastery.

Fractional knowledge and mastery enhance analytical and problem-solving abilities. Fractions encourage the development of critical thought, giving people the tools they need to make wise judgments. This impact goes beyond mathematical computations, infiltrating daily life and making problem-solving possible in various situations (Confrey & Harel, 1994).

Fractions in Various Mathematical Fields

Fractions are crucial in various mathematical disciplines, serving as the foundation for numerous mathematical operations. In addition to fundamental arithmetic, as we stated in previous sections, their significance extends to algebra, geometry, calculus, statistics and probability (Clarke et al., 2008; Devlin, 2008; Tarr et al., 2008).

In the field of geometry, fractions play an indispensable function. They are essential for illustrating the ratio between geometric structure areas, volumes, or lengths, revealing their relative sizes and complex relationships. Using fractions, we can quantify and compare geometric entities, ranging from simple shapes to complex three-dimensional figures. Typically, fractions convey the relationship between a circle's area and radius squared, the ratio of a circle's circumference to its diameter, and the proportional dimensions within similar figures.

Algebra is another field where fractions are applicable. Fractions are frequently used for manipulating algebraic expressions and solving equations. They are used to articulate polynomial division and rational functions, demonstrating their universal applicability and adaptability in various mathematical contexts.

As one advances into calculus, the branch of mathematics concerned with rate change and accumulation of quantities, fractions assume an even greater role. Derivatives and integrals, which are inherently based on the notion of a fraction—a change in quantity over a change in time—are supported by fractions. Fractions play an important role when dealing with complex numbers, allowing us to manipulate fractions of an imaginary unit and opening up an entirely new dimension of mathematical exploration.

In statistics and probability, fractions are essential for communicating the likelihood that an event will occur. Probabilities are frequently articulated as fractions, which define the proportion of positive outcomes to the total number of outcomes. In data analysis, fractions compute mean, median, mode, and even more complex statistical measures like standard deviation and variance.

These diverse applications of fractions in various mathematical disciplines highlight their fundamental nature. Fractions are not merely a discrete mathematical concept within arithmetic. Instead, they provide a deeper comprehension of various mathematical concepts and enable us to accurately model and interpret the world. Fractions elucidate the intricate interrelationships between mathematical concepts and their real-world applications by translating abstract concepts into concrete numbers and calculations.

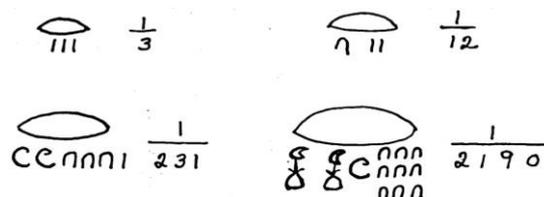


Figure 2. Fractions in ancient Egypt (Source: <https://eliflab.medium.com/egyptian-fractions-a1c75e54deb3>)



Figure 3. Rhind Papyrus (Source: https://www.britishmuseum.org/collection/object/Y_EA10058)

The Historical Growth of Fractions

The Egyptians

Since ancient civilisations such as the Egyptians and Babylonians, fractions have been utilised (Friberg, 2007; Robson, 2008). Figure 2 offers insight into the mathematical methodologies used by ancient Egyptians, highlighting their distinctive approach towards the concept of fractions. The ancient Egyptians had a numerical system that relied on unit fractions, whereby each fraction had a numerator of one. The use of hieroglyphic inscriptions and symbols serves as a visual representation of fractions in some daily activities. Figure 2 emphasises the disparity between the historical approach used by ancient Egyptians and the contemporary practice of using fractional representations, providing a captivating perspective on the progression of mathematical cognition.

As an example, rather than using the fraction 3/4, might express this numerical value as the combination of 1/2 and 1/4. The use of distinctive hieroglyphic writings and symbols to represent fractions was widely seen in a range of publications, notably the renowned *Rhind Mathematical Papyrus* (Figure 3). The papyrus in question exhibits a collection of difficulties and corresponding answers that serve to showcase the Egyptians' adeptness in manipulating fractional values, particularly within the realms of commerce and architecture activities.

Ancient Greece

The Greeks approached fractions from a more philosophical perspective. Renowned mathematicians like Pythagoras (Figure 4) and Euclid study the properties of numbers, with a particular focus on fractions. Their explorations laid the groundwork for modern geometry and number theory. Furthermore, their pioneering work led to the recognition of irrational numbers, which emerged from their in-depth study of

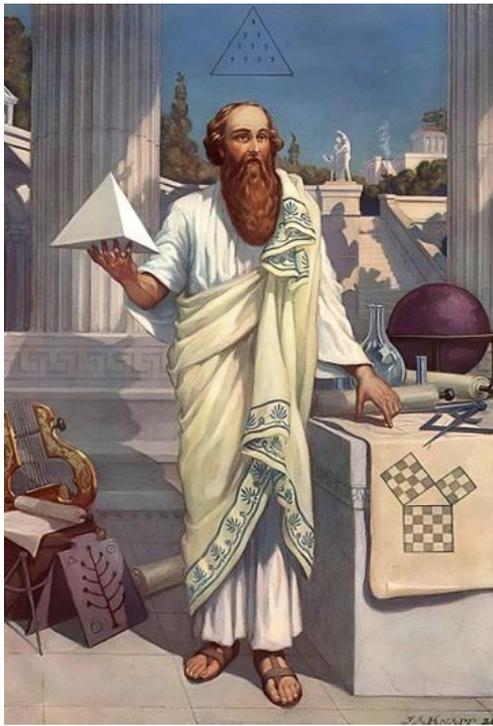


Figure 4. Pythagoras (Source: <https://www.havefunwithhistory.com/facts-about-pythagoras/>)

fractions and ratios. Their contributions have been foundational, shaping the mathematical principles we rely on today (O'Connor & Robertson, 2000).

Medieval Islamic World

The mediaeval Islamic mathematicians played a key role in the preservation and advancement of fractional knowledge. They undertook the task of translating many documents from the Greek language and were instrumental in the introduction of the decimal positional number system. Al-Khwarizmi, a renowned mathematician, authored comprehensive works on the topics of fractions and algorithms, so establishing a foundation for the development of contemporary algebra (Rashed & Armstrong, 1994).

Renaissance Europe

This period saw a significant resurgence in the realm of mathematical sciences. The use of decimal fractions presented a streamlined approach for doing calculations, namely within the domains of commerce and astronomy. During this specific historical period, there was a significant increase in the use of the horizontal fraction bar. This symbol remains prevalent in modern times.

Modern era

Today, fractions are an integral part of our daily lives. From measuring ingredients in cooking to determining interest rates in finance, they are ubiquitous. The advent of computers and calculators has further simplified the

handling of fractions, making complex calculations accessible to everyone.

So, during a long period notable mathematicians contributed to the advancement of fractional knowledge (Bashmakova & Smirnova, 2000; Euclid & Heath, 1956). Based on their substantial contributions, contemporary mathematical principles were developed.

From primordial civilisations to the contemporary era of mathematics, fractions have undergone significant changes. However, their fundamental concept of representing elements of a whole has remained unchanged, highlighting their timeless significance.

Understanding Fractions-Focus on Students Until K-10

For students until K-10, fractions constitute a crucial and difficult conceptual leap from their comprehension of whole numbers. Students' mathematical cognition undergoes a paradigm shift as they progress from the domain of discrete quantities to the more abstract world of fractions. This transition is crucial because it affects a child's understanding of fractions and provides the foundation for future mathematical skills such as algebraic reasoning and the understanding of rational numbers (Confrey & Harel, 1994).

The instructional strategies utilised during this transformational phase profoundly affect students' mathematical knowledge and dispositions. Therefore, a carefully constructed pedagogical strategy is essential for effectively guiding students along this path.

Educators should focus on simplifying fraction concepts and breaking them into readily consumable ideas in the early phases of fraction instruction. Manipulatives such as circular charts or number lines could be utilised to visualise fractions and establish their relationship to whole numbers. These tools offer students concrete representations of the abstract concepts of fractions, bridging their prior comprehension of whole numbers to the new fractional concepts. Nowadays, not only the use of manipulable materials but also the use of digital tools is essential to the student's better understanding.

It is also indispensable to emphasise to students the practical application of fractions. Examples such as dividing a pizza or a cake, sharing chocolates, and measuring ingredients for a recipe can make the study of fractions relevant and engaging for students. Connecting fractions to real-world situations makes the abstract concept friendlier and stresses the practical application of fractions, thereby increasing student motivation and interest in learning.

Instruction should progressively emphasise more abstract and formal fraction operations, such as addition, subtraction, multiplication, and division of fractions. This progression should be scaffolded, building on students' prior knowledge and introducing them

gradually to sophisticated operations. Incorporating problem-solving activities that necessitate the application of fraction operations can also considerably improve students' conceptual understanding and mathematical reasoning abilities.

Consistent formative assessments are crucial for monitoring students' fractions knowledge and providing timely feedback. These assessments can aid in identifying areas in which a student may be underperforming, allowing for the delivery of instruction and interventions specifically tailored to the student's needs.

Comprehending fractions for students until K-10 is a delicate and essential phase of a student's mathematical education. The educators can effectively facilitate this transition, fostering students' mathematical proficiency and appreciation for the subject, by implementing strategic, contemporary, and progressive teaching techniques.

DEVELOPMENTAL CONSIDERATIONS FOR TEACHING FRACTIONS TO STUDENTS UNTIL K-10

Cognitive Growth and Understanding of Fractions Across Age Groups

It is impossible to amplify the impact of cognitive development on children's comprehension of fractions. During the concrete operational stage, notably between the ages of seven and ten, children begin to comprehend mathematical concepts such as fractions, according to Jean Piaget's theory of cognitive development. This stage is characterised by the development of logical thought and comprehension of conservation, which is the realisation that the quantity remains constant even if its appearance changes.

Nonetheless, the timing and rate at which this comprehension develops can be a highly individualised process. It can depend on various factors, such as the child's cognitive development, mathematical aptitude, and exposure to mathematical concepts. Consequently, children's comprehension of fractions can vary considerably.

In early elementary school, children frequently view fractions as straightforward elements of a whole or a particular set. This comprehension corresponds with their cognitive ability to categorise and classify objects based on their shared characteristics, a characteristic of the concrete operational stage. In other words, the concept of fractions as dividing a whole into equal parts is intuitive for children at this age.

As children age and their cognitive capacities develop, so does their understanding of fractions. Students begin to understand that fractions can represent not only a portion of a whole or set but also the

division concept and ratios. They can comprehend fractions as relationships between quantities, extending their mathematical reasoning beyond whole numbers (Empson & Levi, 2011).

Common Obstacles and Misconceptions Children Face When Learning Fractions

Learning fractions is a difficult process frequently accompanied by numerous obstacles and misunderstandings. The 'whole number bias', in which infants incorrectly interpret fraction notation as two distinct whole numbers rather than a single rational number, is one of the significant barriers (Ni & Zhou, 2005). This misconception can significantly hinder a child's ability to execute fraction operations, undermining their fundamental understanding of fractions as a unit.

The over-generalisation of the rule that "larger numbers indicate larger values," which applies to whole numbers, is the source of a second source of confusion. This concept is frequently applied to fractions by children, confusing the relative proportions of fractions. This misconception can hinder a child's ability to compare or recognise equivalent fractions (Vamvakoussi & Vosniadou, 2004).

Not only do these misconceptions affect children's comprehension of fractions, but also their ability to employ this knowledge. The inability to recognise fractions as a distinct entity or to comprehend the relative magnitude of fractions can cause difficulties when children attempt to solve real-world fraction problems. This highlights the significance of resolving misunderstandings in early mathematics education.

Addressing these prevalent misunderstandings in learning process requires thoroughly comprehending their origins and creating targeted teaching strategies. This research paper seeks to identify these pervasive challenges and misconceptions and to suggest effective teaching strategies for addressing them.

Methods for Facilitating and Fostering Children's Understanding of Fractions

Implementing effective teaching strategies is crucial to assisting students in overcoming the obstacles inherent to learning fractions. One such method uses visual representations such as fraction bars and number lines. These aides facilitate children's comprehension of fractions as numbers with magnitude. They provide a visual representation of fractions that corresponds to the concrete operational stage of a child's cognitive development (Fennell & Rowan, 2001).

Using tangible manipulatives, such as fraction blocks or fraction circles, is an additional effective strategy. These provide a tangible representation of the relationship between parts and wholes, which can help children better comprehend fractions. Children can

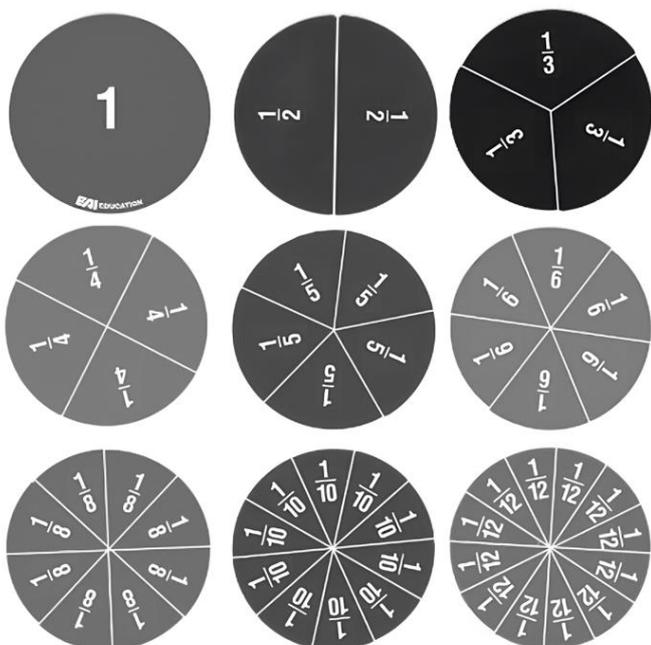


Figure 5. Fraction circles (Source: https://www.eai education.com/Product/533875/Fraction_Circles_Numbered_Set_of_51.aspx)

physically manipulate fractions with manipulatives, strengthening the link between abstract mathematical concepts and their concrete representations.

Integration of fractions into real-world contexts is an additional effective teaching strategy. By presenting fractions in contexts such as cookery or handling money, children can relate the concept to their everyday lives (Carpenter et al., 2003). This makes fractions more relevant and enables children to appreciate the utility of fractions in solving real-world problems.

STRATEGIES FOR TEACHING FRACTIONS FOR STUDENTS UNTIL K-10

Concrete Manipulatives and Visual Materials to Assist Fraction Comprehension

Manipulatives and visual materials are essential instruments in simplifying fractions for children. These hands-on tools, including fraction circles, fraction bars, and pattern blocks, help to change the abstract notion of fractions into a tangible format, something more significant for children, making it easier for learners to grasp the concepts of 'part' and 'whole' (Moseley & Okamoto, 2008). Beyond just understanding, these interactive manipulatives allow students to engage with mathematical representations physically, and these actions will aloud have more interactive and enjoyable learning experiences. Regarding the previous materials, we can give more insights into each one:

- Fraction circles (Figure 5) are particularly useful, with different sections coloured differently to represent different fractions. For instance, a circle

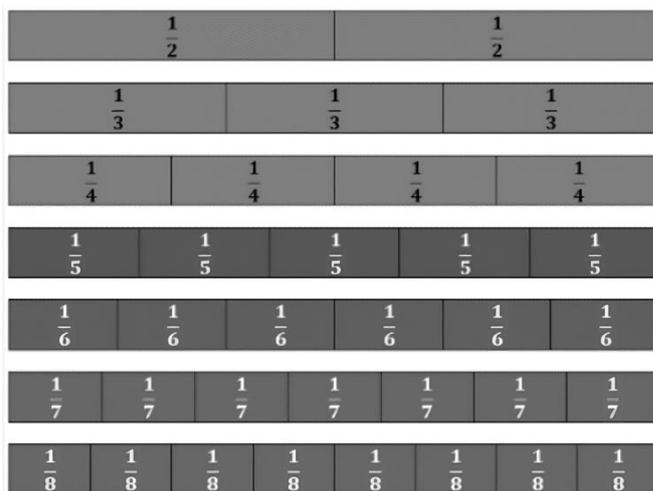


Figure 6. Fraction bars (Source: <https://study.com/learn/lesson/fraction-strip-chart-model-maker.html>)

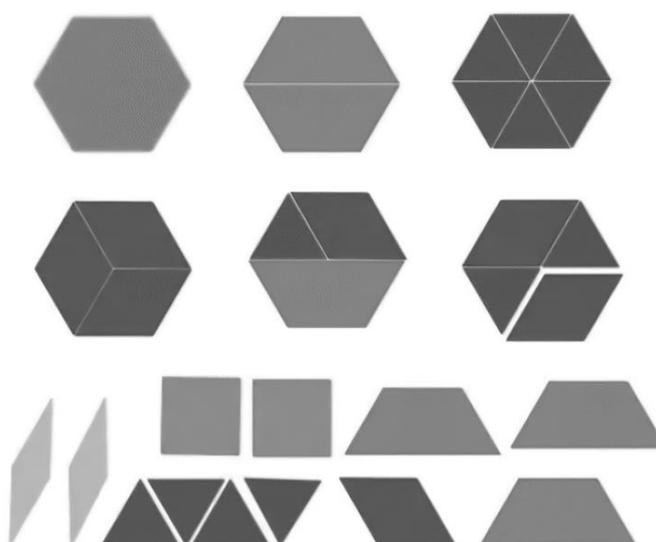


Figure 7. Pattern blocks (Source: <https://www.learning resources.com/item-giant-magnetic-pattern-blocks>)

may be divided into four equal parts, each coloured differently, representing one-fourth of the circle. This allows children to visualise and interact with fractions, giving them a concrete understanding of what it means for something to be a 'part of a whole.'

- Fraction bars or strips (Figure 6) are another form of concrete manipulative. These rectangular bars are divided into equal parts, each representing a fraction of the whole. Like fraction circles, fraction bars make understanding fractions easier as students can see the relative sizes of different fractions, comparing the representations of the fractions with the areas provided by the bars.
- Pattern blocks (Figure 7) are another effective visual tool that can be used to teach fractions. These blocks come in different shapes and sizes, such as triangles, squares, and hexagons, allowing children to visualise and understand the

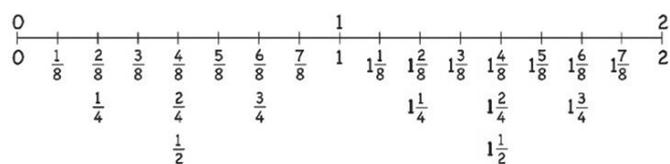


Figure 8. Number line with fractions (Source: <https://www.teachstarter.com/gb/teaching-resource/fractions-number-line-halves-quarters-eighths-gb/>)

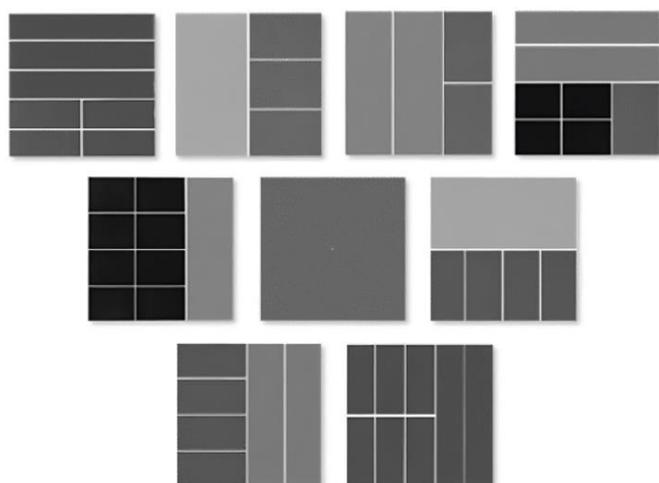


Figure 9. Areas models for fractions (Source: <https://www.k-5mathteachingresources.com/fraction-models.html>)

relationships between different fractions. By experimenting with how different shapes fit together, children can explore equivalent fractions and other fraction concepts hands-on, intuitively.

- Number lines (Figure 8) and area models offer visual help that enhances a child’s understanding of fractions. As visual depictions, they allow students to see and compare the relative sizes and values of different fractions. A number line provides a linear model for visualising fractions, helping children see how one fraction can be larger or smaller than another. It also assists in teaching addition and subtraction of fractions, where children can ‘jump’ along the number line to perform these operations.
- Area models (Figure 9), in contrast, demonstrate how a whole can be divided into equal portions, solidifying the concept of fractions. They can be presented as shapes, like circles or squares, divided into equal parts. Each part can be shaded or filled to represent different fractions. Area models also facilitate the understanding of multiplication and division of fractions, demonstrating these operations straightforwardly and visually.

All these tools embody Bruner’s theory of representation, which suggests that instruction should progress from three different stages:

- Concrete (enactive stage)–This is the stage where the learning takes place through a set of actions. Children interact physically with the environment around them. The knowledge is stored in the form of motor responses. In learning fractions, this could involve using manipulatives like fraction bars or circles, which students can physically handle to understand the relationship between the part and the whole.
- Iconic stage (visual)–This stage involves using visual aids and images to understand concepts. Students might use visual models like number lines or area models to understand fractions in this stage. Visual aids provide a graphical or visual representation of the concept, making it easier for the students to hold the idea.
- Symbolic stage (abstract)–This is where information is stored in the form of a code or symbol, like language. Mathematical symbols and numbers are abstract and understanding them requires the ability to associate these symbols with concrete or visual experiences from the two previous stages described. Regarding the learning of fractions, students at this stage will work directly with numbers and mathematical symbols to perform operations with them.

This gradual approach enables students to build a comprehensive understanding of fractions, thereby equipping them to tackle more complex mathematical challenges in the future.

Additionally, technological advancements have also introduced virtual manipulatives and visual aids. These digital tools, available through various educational software and apps, offer dynamic and interactive ways for children to explore fractions. They can manipulate these virtual tools much like their physical counterparts but with the added advantages of convenience, accessibility, and adaptability to individual learning preferences and paces.

Some examples of digital tools regarding the use of fractions:

- Khan Academy offers a comprehensive range of lessons on fractions, including interactive exercises and instructional videos. The platform allows students to practice independently and provides instant feedback to the student. Website: <https://www.khanacademy.org>
- Number Line by Math Learning Center: This app allows students to visualise number sequences and demonstrate strategies for counting, comparing, adding, subtracting, multiplying, and dividing on a number line. Website: <https://www.mathlearningcenter.org/resources/apps/number-line>

- Fraction Mash: This app allows students to create visual fractions representations using photos on their devices. It provides an engaging and personalised approach to learning about fractions. Website: <https://fractionmash.newvisions.org>
- Woot Math: This platform provides interactive tools for fractions, decimals, and ratios. It offers digital manipulatives like number lines, fraction circles, and grids to help students understand these concepts. Website: <https://www.wootmath.com>
- Brainiaccamp: This platform offers virtual manipulatives that cover several mathematical concepts, including fractions. It's interactive and allows students to manipulate the tools like physical objects. Website: <https://www.brainiaccamp.com>

Therefore, these types of concrete manipulatives, visual aids and digital tools play an indispensable role in mathematics education, particularly in the teaching and learning of fractions. These tools can help educators create engaging and effective learning experiences that cater to different learning styles, thereby promoting a deeper understanding of mathematical concepts among their students.

Use of Real-World Examples and Contexts to Make Fractions Meaningful and Relevant

Including real-world examples and contexts in fractions brings abstract concepts closer to children's everyday lives (Van de Walle et al., 2013). Children can assimilate these concepts through everyday activities such as dividing a pizza, sharing a candy bar, and measuring ingredients for a recipe, which provide concrete examples of fractions.

The incorporation of money and time into fraction teachings can enhance comprehension. When children calculate discounts or acquire the concept of half-price, they indirectly engage in fractional operations. This demystifies the concept of fractions and makes their learning process more engaging and relevant.

This strategy is consistent with the contextual learning theory, which links learning to real-world experiences. When students recognise the practical application of fractions, they are more likely to retain and employ the information. Thus, teaching fractions in context helps reconcile the divide between classroom learning and practical application, enhancing the effectiveness and significance of the learning process.

The combined use of these methods fosters engaging and effective learning, ensuring that children comprehend fractions conceptually and apply them in real-world situations. As we progress into the 21st century, it becomes increasingly important for educators to cultivate these hands-on skills and experiential

learning approaches. They aid in understanding mathematical concepts, like fractions, and equip students with critical skills necessary for future learning and problem-solving.

PROGRESSION OF FRACTIONAL IDEAS AND TECHNIQUES FOR STUDENTS UNTIL K-10

Introduction to Fundamental Fractional Ideas

Introducing straightforward and fundamental concepts such as halves, thirds, and fourths is the first step in teaching fractions. In this initial stage, the instructor presents fractions as divisions of the whole. This concept is readily demonstrated with real-world examples, such as dividing an apple or pizza into equal portions (Watanabe, 2006). This method enables children to intuitively hold the concept of fractions because they can relate to and comprehend them physically.

The effectiveness of instructional tools such as fraction bars and pie charts in illustrating these fundamental concepts can be significantly enhanced. These visual aids assist children in comprehending and visualising fractions. Seeing and manipulating these representations can strengthen students' comprehension of fractions as parts of a whole (Boulet, 2017).

As students begin dominating these fundamental concepts, they are taught to recognise fractions as numbers in their own right. Students are taught, for instance, to locate a fraction on a number line, reinforcing the notion that fractions are numbers with specific positions and magnitudes (Siegler et al., 2011).

Overview of Fraction Operations

In the next phase, students are introduced to more complex operations involving fractions, starting with addition and subtraction. This stage builds upon the foundational understanding of basic fraction concepts, ensuring a continuous link between new knowledge and prior learning. To demonstrate the procedures of adding and subtracting fractions with identical denominators, educators can leverage visual aids such as fraction bars, simplifying the learning process.

As students gain proficiency in basic fraction operations with identical denominators, they are gradually exposed to operations involving fractions with different denominators. This step-by-step progression helps students understand the requirement for a common denominator and the methods to determine one. As this phase allows students to grasp more intricate fraction operations, it plays a crucial role in achieving fluency in fractions (Siegler et al., 2011).

Acquiring Competences in Comparison, Equivalence, and Conversion

The final phase of fraction instruction focuses on developing fraction comparison, equivalence, and conversion skills. Teachers can facilitate the work done by the students on comparing fractions with different denominators using visual assistance such as fraction bars and number lines. Children can identify the larger or smaller fraction through visual comparisons, enhancing their understanding of the relationships between fractions (Siegler et al., 2011).

Diagrams and manipulatives also play a significant role in assisting students to comprehend the concept of equivalent fractions. Students understand how different fractions can represent the same quantity by visualising and manipulating the same fraction represented in different ways (Pantziara & Philippou, 2012).

Another important aspect is the developing skills for converting between fractions, decimals, and percentages, which is a crucial aspect of fractions instruction. This skill is a foundation for further mathematical applications, making it an essential component of understanding fractions (Hiebert & Grouws, 2007).

Teaching fractions is a deep process that requires a methodical and sequential process. Each new concept introduced expands upon the knowledge acquired in earlier phases. That is one of the aspects that is well-known in mathematics. Each step is meticulously designed to ensure a structured approach, from the simple division of a whole into fractions to complex operations and comparisons. This strategy gives students a solid mathematical foundation, paving the way for their continued growth and development.

COMPREHENSIVE ASSESSMENT OF FRACTION COMPREHENSION FOR STUDENTS UNTIL K-10

A thorough evaluation of a student's comprehension of fractions is essential for improving mathematics education for students in grades until K-10. The multiple assessment strategies teachers can use are a path for the multidimensional view of students' fraction comprehension and practical application. These strategies can be split into formative assessments, performance-based tasks, tests of conceptual understanding, and technology-enhanced assessments.

Formative Evaluations

The continuous formative assessments provide real-time insight into a student's fractional understanding. These evaluations could include fraction problem-solving short tests, explanations of fractions, and illustrative tasks such as dividing a figure into equal parts (Black & Wiliam, 2009). Implementing these

assessments regularly will ensure that educators can monitor and improve their teaching methods while, at the same time, also giving the students a better understanding of their learning progression. Moreover, they enable students to reflect on their understanding and learn from their errors in a non-threatening setting.

The feedback is an essential component of formative assessments because it allows teachers and students to resolve misconceptions or gaps in understanding before moving on to further steps. In the context of fractions, this feedback could consist of rectifying a student's misconception about why fractions with differing denominators cannot be directly compared or encouraging students to elaborate on their thought processes when solving fraction problems. Teachers can ensure that their instruction meets the students' requirements and adapt as necessary by incorporating these assessments into their daily routines.

Activity-Based Tasks

Performance-based assignments offer a practical perspective on fraction learning by emphasising the implementation of learned concepts in real-world scenarios. Students could engage in activities such as dividing a group of objects into fractions or measuring quantities with fractions. This approach simulates real-world situations and bridges the divide between the abstract world of fractions and daily life (Moseley & Okamoto, 2008).

Performance-based tasks enable students to demonstrate their knowledge in a more authentic setting than the traditional paper-and-pencil exams. In addition, it can motivate students to engage in mathematical reasoning and problem-solving activities, two of the major essential skills for more advanced mathematical work in the classroom. Teachers can observe students' strategies and thought processes as they complete these assignments, providing another valuable source of information about their comprehension of these activities.

Tests of Conceptual Understanding

Using conventional assessments, Cai (1995) stated that performance-based assignments offer a new path for evaluating students' understanding of fractions. These assignments, as explained by Daro et al. (2011), complement traditional tests by shifting the focus from abstract comprehension to the practical application of mathematical concepts, particularly fractions, in the real-world situations already discussed previously in this section.

Such tasks can be diverse and multifaceted. For instance, as Lesh et al. (2000) proposed, students might be asked to adjust recipe measurements to serve a different number of people, thereby applying their understanding of fraction multiplication or division.

Alternatively, students could be tasked with determining the amount of paint needed to cover a certain proportion of a wall, thereby applying fractions to an area problem.

Performance-based assignments also offer a unique opportunity to evaluate students' mathematical communication and reasoning skills. According to NCTM (2000), students should be encouraged to explain their problem-solving process, justify their solutions, and respond to the solutions of others. For example, a student might be asked to demonstrate how to divide a cake into equal parts and then explain why their method ensures everyone gets an equal share.

This approach facilitates differentiated instruction and learning, as Tomlinson (2001) emphasised. By observing how a student tackles a performance-based assignment, teachers can gain valuable insights into a student's strengths and areas needing improvement. These insights can tailor future instruction, provide targeted support, and create an individualised learning plan.

While the traditional tests, examined by Cai (1995), remain an important tool for assessing conceptual comprehension, performance-based assignments provide a more nuanced understanding of a student's fraction knowledge and application. As Boaler (1998) noted, these assignments highlight the importance of fractions in everyday scenarios, showing the relevance and practicality of what students learn in the classroom. Hence, combining traditional and performance-based assessments, as Shepard (2000) suggested, allows for a more robust and comprehensive evaluation of a student's understanding of fractions.

Technology-Enhanced Evaluations

Regarding the work of Quellmalz et al. (2009), integrating technology in educational assessment brings an additional dimension of engagement and interactivity to students' exploration of fractions. Educational applications and online platforms frequently feature tools and manipulatives that allow students to understand the principles underlying fractions visually and interactively. An intuitive knowledge of the relationships between the parts and the total of fractions can be attained, for instance, through virtual number lines or pie charts.

Moreover, incorporating gaming elements within these platforms, such as rewards, levels, or missions, can turn learning fractions into a fun and immersive experience. Gamification is also something that could be used in the teaching of mathematics. By framing mathematical problems within a game or story context, students can be more motivated to persist and succeed. This is consistent with researchers like (Kiili et al., 2012), who found that game-based learning can enhance

student engagement and motivation, like the gamification stated before.

A significant advantage of technology-enhanced assessments, especially when discussing mathematical fractions, is their capacity for personalisation and adapting to specific situations. These systems can use adaptive learning algorithms to adjust the complexity of problems based on the student's performance (Pardos et al., 2013). This ensures that students are neither overwhelmed by excessively challenging fraction problems nor bored with basic ones, such as simplifying $\frac{1}{2}$ or determining $\frac{3}{4}$ of a given number. Furthermore, AI-driven systems can provide real-time feedback, allowing students to understand their mistakes immediately, such as incorrect numerator or denominator placements, and adjust their learning strategies accordingly (Heffernan & Heffernan, 2014). Such systems also have the potential to identify and address misconceptions specific to fractions, ensuring a deeper understanding of the subject (Roll & Wylie, 2016). A good example is when students often struggle with the concept of equivalent fractions or adding fractions with different denominators. AI can detect these specific challenges and provide targeted exercises to address them. As AI continues to evolve, its integration into educational platforms promises much more tailored and practical learning experiences in fractions, bridging the gap between individual needs and curriculum demands.

But the power of using the AI can be also the instant feedback that can be offered, helping the students to understand their mistakes and learn immediately from them, reinforcing correct responses and discouraging misconceptions. As Shute (2008) found, this timely and targeted feedback can significantly enhance learning.

A holistic and multi-faceted assessment approach is necessary when considering the student's mastery of fractions. As underlined by Cai (1995) and Quellmalz et al. (2009), this strategy should encompass formative assessments, performance-based tasks, tests of conceptual comprehension, and technology-enhanced assessments. Each type of assessment provides different insights and presents a well-rounded view of the student's understanding.

Such an approach promotes personalised instruction, allowing teachers to adapt their teaching methods to individual student's needs and preferences, an approach encouraged by Tomlinson (2001). This enhances the learning experience and increases the likelihood of mastering fractions, ensuring the instruction matches each student's learning pace.

Therefore, in the era of digital learning, integrating technology in assessing students' understanding of fractions offers a wealth of possibilities to engage students, personalise instruction, and provide immediate feedback. However, combining it with traditional assessment methods will be important

because it contributes to a comprehensive, effective, and engaging learning experience.

CONCLUSION AND FUTURE CONSIDERATIONS

Understanding and mastering fractions is essential for the mathematical development of students. As described in the previous sections, this document begins with an investigation of fractions using concrete and pictorial representations, which are effective for the initial introduction and comprehension of the fraction concept. The analysis progressively advances to abstract numerical expressions, consolidating students' conceptual understanding. Then, we introduced fundamental fraction concepts, complex operations, fraction comparison, equivalence, and conversion skills. Throughout these phases, the consistent use of manipulatives, visual aids, and real-world examples was emphasised as a significant learning support to extend the students' knowledge. The teaching progression highlighted the significance of interconnectedness by emphasising scaffolded learning and expanding on previous knowledge. Like other mathematics areas, the mathematical foundations of fractions need a comprehensive and supportive basis well-structured from an early age.

We stated in this article the need for a comprehensive evaluation of the student's understanding of fractions using various assessment strategies, including formative assessments, performance-based tasks, measures of conceptual understanding, and technology-enhanced assessments.

These elements suggest a structured and systematic approach to teaching fractions, complementing various assessment strategies. However, the emergence of technology, especially artificial intelligence (AI), in recent years suggests optimistic future developments in fraction learning and other areas of mathematics. AI's ability to adapt to individual learning patterns and provide personalised learning experiences can revolutionise the teaching and learning of fractions and many other mathematical concepts (Baker & Inventado, 2014). Integrating AI-driven tools in the classroom can offer students a more tailored and engaging learning experience, addressing their unique needs and challenges (Luckin et al., 2016). Furthermore, some gamification processes teachers use inside the school environment will soon become even more standard as a teaching and assessment tool. Experience shows that games and numerous digital tools are successfully used in schools for gamification (Vrcelj, 2023).

The AI's ability to adapt to individual learning patterns and provide personalised learning experiences can revolutionise the teaching and learning of fractions and many more concepts in mathematics. For example, AI-powered educational platforms can dynamically

alter the difficulty level of fraction assignments based on previous student performance, resulting in a more personalised and engaging learning experience, opening several paths according to the growth of students' knowledge, providing real-time feedback on students' progress, identify areas of difficulty, and even recommend specific intervention strategies to enhance performance. Future advancements in AI may also include more sophisticated modelling of fraction concepts and virtual and augmented reality experiences that enable immersive and intuitive fraction exploration.

In conclusion, while the traditional structured, systematic, and multifaceted approach to teaching fractions provides a solid foundation for student learning, incorporating more digital tools, like, for instance, AI, could transform the learning experience for the teacher and obviously for the student. Continuous research and development of AI-powered educational tools will undoubtedly pave the way for innovative, efficient, and pleasurable fraction learning, something that, after all these years, continues to be a heavy burden in the arms of the teachers and minds of the students.

We emphasise the importance of using new technology in education besides the traditional structured tools so that teachers serve as facilitators and students take the lead in their mathematical learning as investigators (Hwang et al., 2023).

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